**CFT TALK**

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# 1 Classical introduction

- Let $K$ be a number field, $\mathcal{O}_K$ be its ring of integers, $\mathfrak{m}$ a modulus containing the infinite primes, and $\chi : \mathbb{I}^{\mathfrak{m}}/\mathbb{H}^{\mathfrak{m}} \to \mathbb{C}^\times$ be a character. Extend $\chi$ to all ideals by 0.
- Define for $s$ in a suitable right halfplane

$$L(s,\chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{|N(\mathfrak{a})|^s}$$
$$\zeta_K(s) = \sum_{\mathfrak{a}} |N(\mathfrak{a})|^{-s}.$$

- A central result is

**Theorem 1.** Let $r_1$ be the number of real embeddings of $K$, $r_2$ the number of complex embeddings. Let $\Delta_K$ be the discriminant of $K$. Define
$$\Lambda_K(s) = |\Delta_K|^{s/2}(\pi^{-s/2}\Gamma(s/2))^{r_1}(2(2\pi)^{-s}\Gamma(s))\zeta_K(s).$$
Then $\Lambda_K$ has a meromorphic continuation to the whole complex plane, satisfying the functional equation
$$\Lambda_K(s) = \Lambda_K(1-s).$$
$\zeta_K$ has a simple pole at 1, with residue
$$\lim_{s\to 1}(s-1)\zeta_K = \frac{2^{r_1}(2\pi)^{r_2}h_K}{w_k\sqrt{|\Delta_K|}} \cdot \mathrm{Reg}_K$$

- Classically, one proves the analytic continuation and functional equation using Hecke's extension of Riemann's argument: From the identity
$$y^{-s}\Gamma(s) = \int_0^\infty u^{-s}e^{-uy}\frac{\mathrm{d}u}{u}$$
compute
$$\zeta_K(s) = \sum_{\mathfrak{a}} |N(\mathfrak{a})|^{-s} = \frac{\int_0^\infty u^{-s}\sum_{\mathfrak{a}} e^{-u|N(\mathfrak{a})|}\frac{\mathrm{d}u}{u}}{\Gamma(s)}$$
Speaking very roughly, integral ideals correspond to points on some free $\mathbb{Z}$ module, and the function $u \mapsto \theta_K(u) = \sum_{\mathfrak{a}} e^{-u|N(\mathfrak{a})|}$ is essentially a weighted sum over that $\mathbb{Z}$ module.
- Break the integral $\int_0^\infty u^{-s}\sum_{\mathfrak{a}} e^{-u|N(\mathfrak{a})|}\frac{\mathrm{d}u}{u}$ into two parts,
$$(1) = \int_1^\infty u^{-s}\sum_{\mathfrak{a}}\theta_K(u)\frac{\mathrm{d}u}{u}$$
$$(2) = \int_0^1 u^{-s}\theta_K(u)\frac{\mathrm{d}u}{u}$$
- (1) converges rapidly, absolutely, at both 1 and $\infty$ for all $s \in \mathbb{C}$ and thus defines an entire function.

- (2) has issues at 0. Changing variables $u \mapsto u^{-1}$, (2) transforms
$$(2) = \int_1^\infty u^s \theta_K(u^{-1}) \frac{\mathrm{d}u}{u}.$$
- To proceed, we need to relate $\theta_K(u^{-1})$ with $\theta_K(u)$. The essential fact, is that $\Theta_K$ is a prototype of a class of functions that has a rich group of symmetries: modular forms. Proving this requires some Fourier analysis, and the precise transformation law of $\theta_K$ is delicate.
- Nonetheless, the transformation law gives rise to an expression for (2) which:
  - is nicely convergent at 1 and $\infty$ for all $s$ away from 1
  - gives rise to those nutty factors in the functional equation
  - is symmetric with the expression (1) under the transformation $s \mapsto 1 - s$.

# 2 Gripes, desiderata

- You can't throw a punch at $\zeta_K$ without something number theoretical, so what on earth is that funky factor appearing in the functional equation?!??!
- H.M. Stark reportedly said:

  > The zeta-function of a field is like the atom of physics. (. . .) we will show how to split it via group theory

  Which refers to how $\zeta$ functions split as products of certain $L$ functions attached to certain groups which store arithmetical data. If our main interest is in understanding $\zeta$, then we should be focusing our attention on $L$.
- This proof makes no real use of the Euler product of $\zeta_K$ which is really where all the number theory.
- J. Tate, in his 1950's graduate thesis under E. Artin (Tate made essential use of M. Matchett's, a former student of Artin, work on the topology of the a/ideles). Gave a local description of $L$ functions which describes completely the factors appearing in the functional equation. One should also note that Iwasawa developed an analogous theory almost simultaneously, indepenently. This can be found in his published letter to Dieudonne.

# 3 Modern Tate–Matchett–Iwasawa Setup

## 3.1 Preliminary global stuff

- Prof. Goins gave a description of the adeles and ideles as sets:

$$\mathbb{A} = \text{infinite 'vectors' } (\alpha_\nu) \text{ indexed by places, where } \alpha_\nu \in \mathcal{O}_\nu \text{ for all but finitely many } \nu$$
$$\mathbb{I} = \text{infinite 'vectors' } (\alpha_\nu) \text{ indexed by places, where } 0 \neq \alpha_\nu \in \mathcal{O}_\nu^\times \text{ for all but finitely many } \nu.$$

- $\mathbb{A}$ and $\mathbb{I}$ are a locally compact hausdorff topological ring, and group respectively. Their topology is most naturally (and canonically! and uniquely!) determined when $\mathbb{A}$ and $\mathbb{I}$ are viewed as a colimit.
- For any finite set of primes $S$ including the infinite ones, define
$$\mathbb{A}^S = \prod_{\nu \in S} K_\nu \times \prod_{\nu \notin S} \mathcal{O}_\nu$$
$$\mathbb{I}^S = \prod_{\nu \in S} K_\nu^\times \times \prod_{\nu \notin S} \mathcal{O}_\nu^\times$$
- With the product topology, these are locally compact since $\mathcal{O}_\nu$ and $\mathcal{O}_\nu^\times$ are compact and $K_\nu$, $K_\nu^\times$ are locally compact (exercise in any intro topology course).

- The collection of subsets of primes containing the infinite ones are ordered by inclusion with the set, $S_\infty$ containing only the infinite primes, sitting at the base. If $S \subset T$ then there is a continuous, open inclusion $\mathbb{A}^S \to \mathbb{A}^T$, resp $\mathbb{I}^S \to \mathbb{I}^T$.
- Define $\mathbb{A}$ and $\mathbb{I}$ to be the colimit of these diagrams.
- This tells us how to prove a function on $\mathbb{A}$ or $\mathbb{I}$ is continuous: iff it is continuous on every $\mathbb{A}^S$ resp $I^S$.
- The critical point is that to prove something globally about continuity of a function, finiteness/convergence of an integral, etc. it suffices to prove it everywhere locally–permitting finitely many 'bad' primes.
- $\mathbb{A}$ and $\mathbb{I}$ are locally compact abelian hausdorff groups, so they are equipped with translation invariant integrals given by $\mathrm{d}x$ and $\mathrm{d}^\times x$, unique up to scale. These measures will be clarified by working locally.

## 3.2 Local setup

- Now let $k$ be a local field with valuation $\nu$, so $k$ is a finite extension of $\mathbb{Q}_p$ where $\nu | p$ for some $p$, $\mathbb{R}$, or $\mathbb{C}$.
- Local fields are self dual, so to determine their character group, it suffices to determine a single nontrivial character. We construct one now for each type of $\nu$. First an auxiliary construction:
  - if $\nu$ is $\mathfrak{p}$-adic, for $x \in \mathbb{Q}_p$ define $\lambda(x)$ to be an rational number in $\mathbb{Z}[p]$ such that $\nu(\lambda(x) - x) \leq 1$.
  - if $\nu$ is archimedean, then $\lambda(x) \in \mathbb{R}$ satisfies $\lambda(x) = -x \mod 1$.
- Let tr be the map defined by summing over galois conjugates. if $\nu$ is real, then this is just the identity, if $\nu$ is complex this is just $2\,\mathrm{Re}(x)$.
- Then define a character for any $\nu$

$$x \mapsto e^{2\pi i \lambda(\mathrm{tr}\, x)}.$$

- This is continuous, and the isomorphism $k$ to its dual is given by

$$\xi \mapsto (x \mapsto e^{2\pi i \lambda(\mathrm{tr}\, x\xi})$$

- We normalize Haar measures as follows:
  - If $\nu$ is $\mathfrak{p}$-adic, require $\int_{\mathcal{O}_\nu} 1 \,\mathrm{d}x = (N\delta)^{-1/2}$ where $\delta$ is the different of $k$.
  - If $\nu$ is real, then $\mathrm{d}x$ is the Lebesgue measure.
  - If $\nu$ is complex, then $\mathrm{d}x$ is two times the lebesgue measure in the plane.