ZETA FUNCTIONS OF REAL QUADRATIC FIELDS AS PERIODS OF EISENSTEIN SERIES

JUSTIN KATZ

1. Introduction

Set $k = \mathbb{Q}(\sqrt{D})$ and $\mathcal{O}_k$ its the ring of integers. The zeta function attached to k is

$$\zeta_k(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_k} \frac{1}{|N(\mathfrak{a})|^{-s}}$$

where the sum is over nonzero integral ideals in $\mathcal{O}_k$ and $\mathrm{Re}(s) > 1$. A suitable modification of Riemann's argument for the continuation of $\zeta = \zeta_{\mathbb{Q}}$ shows that ζ_k has meromorphic continuation to the entire s-plane.

Let $G = \mathrm{SL}_2(\mathbb{R})$, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, and P be the parabolic of upper triangular elements of $\mathrm{SL}_2(\mathbb{R})$. As usual, G acts on the upper half plane $\mathfrak{H}$ by fractional linear transformations. The for complex s with $\mathrm{Re}(s) > 1$, the s^{th} Eisenstein series on the upper half plane is

$$E_s(z) = \sum_{\gamma \in (P \cap \Gamma) \backslash \Gamma} (\mathrm{Im}(\gamma z))^s,$$

which is Γ-invariant by design. For fixed z, the map $s \mapsto E_s(z)$ has meromorphic continuation to the entire s-plane.

This writeup shows that ζ_k is a sum of integrals of Eisenstein series over closed geodesics.

2. Real quadratic

In this section $k = \mathbb{Q}(\sqrt{D})$ with $D < 0$. As a $\mathbb{Q}$ module, k is $\mathbb{Q}^2$. The multiplicative subgroup $k^\times$ acts transitively on $\mathbb{Q}^2$. Choose the basis $\{\sqrt{D}, 1\}$ and compute for $a + b\sqrt{D} \in k^\times$,

$$(a + b\sqrt{D}) \times \sqrt{D} = a \cdot \sqrt{D} + bD \cdot 1$$
$$(a + b\sqrt{D}) \times 1 = b \cdot \sqrt{D} + a \cdot 1.$$

In coordinates, we have an embedding

$$k^\times \to \mathrm{GL}_2(\mathbb{Q})$$
$$a + b\sqrt{D} \mapsto \begin{pmatrix} a & bD \\ b & a \end{pmatrix}$$

Let G' be the image of $k^\times$ in $\mathrm{GL}_2(\mathbb{Q})$. Note that the determinant of the image of $a + b\sqrt{D}$ is $a^2 - b^2 D = N(a + b\sqrt{D})$. The existence of nontrivial units in $\mathcal{O}_k$ implies that the subgroup

$H' = G' \cap G$ is nontrivial. As a subgroup of $G = \mathrm{SL}_2(\mathbb{R})$, the group H' sensibly acts on the upper half-plane. Taking the trace of a generic matrix $Tr\begin{pmatrix} a & bD \\ b & a \end{pmatrix} = 2a$ and recalling that the units of $\mathcal{O}_k$ are integral shows that all nonidentity elements of H_1 are hyperbolic. As such, any nonidentity element of H' fixes two distinct points on $\mathbb{R} \cup \{\infty\}$.

Although H' is discrete in G, it lies in a one parameter subgroup H of G' defined by parameterizing the solutions of $a^2 - b^2D = 1$ viz

$$H' \subset H = \{\begin{pmatrix} \cosh(t) & \sinh(t)\sqrt{D} \\ \sinh(t)/\sqrt{D} & \cosh(t) \end{pmatrix} : \quad t \in \mathbb{R}\}$$

In fact, $H' = H \cap \Gamma$. Denote an element of H by h_t. One can compute that each h_t fixes $-\sqrt{D}$ and $\sqrt{D}$. Consequently, H fixes the geodesic $\mathcal{C}_{\sqrt{D}}$ running from $-\sqrt{D}$ to $\sqrt{D}$, set-wise. In particular, the radius of the semicircle defining $\mathcal{C}_{\sqrt{D}}$ is $\sqrt{D}$, so the point $i\sqrt{D} \in \mathcal{C}_{\sqrt{D}}$. Pointwise, each h_t translates a point rightward along $\mathcal{C}_{\sqrt{D}}$. The orbit of $i\sqrt{D}$ under the nontrivial discrete subgroup $H' = H \cap \Gamma$ partitions the orbit $H \cdot i\sqrt{D}$ into congruent intervals. The resulting quotient $H' \backslash H \cdot i\sqrt{D}$ is compact. Paul Garrett showed for a real quadratic field with principal integer ring that

$$2^s \frac{\sqrt{D}^s \Gamma(\frac{s}{2})\Gamma(\frac{s}{2})}{\Gamma(s)} \frac{\zeta_k(s)}{\zeta(s)} = \int_{H' \backslash H} E_s(h \cdot i\sqrt{D}) dh.$$

The fact that the right side of the above is a single integral amounts to the principality of the ring of integers.

Suppose $\mathcal{O}_k$ is not principal. As with imaginary quadratic fields, each class of ideals, equivalent under multiplication by $k^\times$, corresponds to a class of quadratic forms with discriminant D, equivalent under change of basis by $\mathrm{SL}_2(\mathbb{Z})$. Unlike imaginary quadratic fields, these quadratic forms are indefinite. Thus, the sum over ideals in ζ_k is equivalently a sum over quadratic forms. As we can decompose the sum over ideals into a sum over classes, then over representatives, we can do the same with quadratic forms.

Let $Q(m,n) = Am^2 + Bmn + Cn^2$ of discriminant D. Dehomogenizing the quadratic form gives a polynomial $Az^2 + Bz + C$ which has real roots $\frac{-B \pm \sqrt{D}}{2A}$. Without loss of generality let $\frac{-B-\sqrt{D}}{2A} < \frac{-B+\sqrt{D}}{2A}$ and call the former α. We will find that integrating $E_s(- \cdot \alpha)$ over a conjugate of H will yield the desired sum over the class of quadratic forms containing Q.

The transformation $a = \left[\begin{smallmatrix} 2A & -B \\ 0 & 1 \end{smallmatrix}\right] \in \mathrm{GL}_2(\mathbb{Q})$ takes α to $-\sqrt{D}$ and $\bar{\alpha}$ to $\sqrt{D}$. The whole group H fixes $\pm\sqrt{D}$, so the conjugate

$$H_\alpha = a^{-1} H a = \begin{bmatrix} \cosh(t) - B\sinh(t)/\sqrt{D} & -2C\sinh(t)/\sqrt{D} \\ 2A\sinh(t)/\sqrt{D} & \cosh(t) + B\sinh(t)/\sqrt{D} \end{bmatrix}$$

fixes α and $\bar{\alpha}$. As above, H_α setwise fixes the geodesic $\mathcal{C}_\alpha$ running from α to $\bar{\alpha}$, and pointwise shifts to the right. The geodesic $\mathcal{C}_\alpha$ has radius $(\alpha - \bar{\alpha})/2 = \frac{\sqrt{D}}{2A}$ and is centered about $(\alpha + \bar{\alpha})/2 = -\frac{B}{2A}$, so the point $z_\alpha = \frac{-B + i\sqrt{D}}{2A}$ is on $\mathcal{C}_\alpha$. The discrete subgroup

$H'_\alpha = H_\alpha \cap \mathrm{SL}_2(\mathbb{Z})$ partitions the geodesic $\mathcal{C}_\alpha = H_\alpha z_\alpha$ into compact intervals, so the quotient $H'_\alpha \backslash H_\alpha z_\alpha$ is compact. Now the integral of the Eisenstein series decomposes

$$\begin{aligned}\int_{H'_\mathfrak{a}\backslash H_\mathfrak{a}} E_s(h_\mathfrak{a}\cdot z_\alpha)dh_\mathfrak{a} &= \int_{H'_\mathfrak{a}\backslash H_\mathfrak{a}} \sum_{\gamma\in P\cap\Gamma\backslash\Gamma} \mathrm{Im}(\gamma h_\mathfrak{a}\cdot z_\alpha)dh_\mathfrak{a} \\ &= \sum_{x\in\Gamma\cap P\backslash\Gamma/H'_\alpha} \int_{H'_\mathfrak{a}\backslash H_\mathfrak{a}} \sum_{y\in(x^{-1}(\Gamma\cap P)x\cap H_\alpha)\backslash H'_\alpha} \mathrm{Im}(xyh_\mathfrak{a}\cdot z_\alpha)dh_\mathfrak{a}.\end{aligned}$$

by considering which $y, y' \in$ make $(\Gamma\cap P)x(H'_\alpha)y = (\Gamma\cap P)x(H'_\alpha)y'$. The integral parabolic has eigenvalues ± 1. The eigenvalues of H are nontrivial units (this follows from the definition of the embedding of $k^\times$, or an easy calculation), and H_α is conjugate to H so it has the same eigenvalues. Thus, the intersection in the inner sum is $\{\pm\mathrm{id}\}$. Accordingly, the integral unwinds

$$\int_{H'_\mathfrak{a}\backslash H_\mathfrak{a}} E_s(h_\mathfrak{a}\cdot z_\alpha)dh_\mathfrak{a} = \sum_{x\in\Gamma\cap P\backslash\Gamma/H'_\alpha} \int_{\{\pm\mathrm{id}\}\backslash H_\mathfrak{a}} \mathrm{Im}(xh_\mathfrak{a}\cdot z_\alpha)dh_\mathfrak{a}.$$

Recall that modulo the integral parabolic, Γ is coprime bottom rows. Dividing by $2\zeta(s)$ gives as usual

$$\int_{H'_\mathfrak{a}\backslash H_\mathfrak{a}} E_s(h_\mathfrak{a}\cdot z_\alpha)dh_\mathfrak{a} = \frac{1}{2\zeta(2s)} \sum_{\{(m,n)\neq 0\}/H'_\alpha} \int_{\{\pm\mathrm{id}\}\backslash H_\mathfrak{a}} \frac{\mathrm{Im}(h_\mathfrak{a}z_\alpha)}{|mh_\mathfrak{a}z_\alpha+n|}dh_\mathfrak{a}.$$

For brevity, let $u = \cosh t$ and $v = \sinh t$. Taking the imaginary part in the numerator amounts to multiplying the denominator by the action of the lower row, as usual. The denominator is

$$\begin{aligned}&|m((u-Bv/\sqrt{D})z_\alpha - 2Cv/\sqrt{D}) + n(2Avz_\alpha/\sqrt{D} + u + Bv/\sqrt{D})|^{2s} \\ &= |(m(u-Bv/\sqrt{D}) + 2Anv/\sqrt{D})\frac{-B}{2A} - 2Cmv/\sqrt{D} + n(u+Bv/\sqrt{D}) \\ &+ \frac{i\sqrt{D}}{2A}(m(u-Bv/\sqrt{D}) + 2Anv/\sqrt{D})|^{2s} \\ &= |\frac{-m}{2A}(Bu-\sqrt{D}v) + nu + i(\frac{m}{2A}(u\sqrt{D}-Bv)+nv)|^{2s}\end{aligned}$$

Taking the absolute value,

$$\begin{aligned}(\frac{m}{2A}(\sqrt{D}v-Bu)+nu)^2 + (\frac{m}{2A}(u\sqrt{D}-Bv)+nv)^2 = {}& \frac{m^2}{4A^2}(D(u^2+v^2) - 4B\sqrt{D}vu + B^2(v^2+u^2)) \\ &- \frac{mn}{A}(Bu^2 - 2\sqrt{D}uv + Bv^2) \\ &+ n^2(u^2+v^2)\end{aligned}$$

Recall that $u^2+v^2=\cosh^2 t+\sinh^2 t=(e^{2t}+e^{-2t})/4$ and $uv=(e^{2t}-e^{2t})/4$ so

$$\begin{aligned}(\frac{m}{2A}(\sqrt{D}v-Bu)+nu)^2+(\frac{m}{2A}(u\sqrt{D}-Bv)+nv)^2 &= 2(\frac{D-2B\sqrt{D}+B^2}{4A^2}m^2-\frac{B-\sqrt{D}}{A}mn+n^2)e^{2t}\\ &+2(\frac{D+2B\sqrt{D}+B^2}{4A^2}m^2-\frac{B+\sqrt{D}}{A}mn+n^2)e^{-2t}\\ &=2(\alpha^2m^2+2\alpha mn+n^2)e^{2t}\\ &+2(\overline{\alpha}^2m^2+2\overline{\alpha}mn+n^2)e^{-2t}\end{aligned}$$

Where $\alpha=\frac{-B-\sqrt{D}}{2A}$. To summarize, each integrand in the sum is

$$\frac{(\operatorname{Im} h_{\mathfrak{a}}z_\alpha)^s}{|mh_{\mathfrak{a}}z_\alpha+n|^{2s}}=2^{-s}\frac{\sqrt{D}^s/(2A)^s}{((\alpha m+n)^2e^{2t}+(\overline{\alpha}m+n)^2e^{-2t})^s}.$$

Thus, the integral to compute is

$$\int_{-\infty}^{\infty}\frac{dt}{(Xe^{2t}+Ye^{-2t})^s}.$$

Following Paul Garrett, recall

$$y^{-s}\Gamma(s)=y^{-s}\int_0^\infty u^se^{-u}\frac{du}{u}=\int_0^{-\infty}u^se^{-u\cdot y}\frac{du}{u},$$

specifically:

$$\Gamma(s)\int_{-\infty}^{\infty}\frac{dt}{(Xe^{2t}+Ye^{-2t})^s}=\int_{-\infty}^{\infty}\int_0^\infty u^se^{-u(Xe^{2t}+Ye^{-2t})}\frac{du}{u}dt.$$

Change to multiplicative coordinates in the outer integral via $t=\frac{\log v}{2}$, and then change u to uv, making the above

$$\frac{1}{2}\int_{-\infty}^{\infty}\int_0^\infty u^se^{-u(Xv+Yv^{-1})}\frac{du}{u}\frac{dv}{v}=\frac{1}{2}\int_{-\infty}^{\infty}\int_0^\infty u^sv^se^{-u(Xv^2+Y)}\frac{du}{u}\frac{dv}{v}.$$

Now change v to $\sqrt{v}$, then v by uv so

$$\frac{1}{4}\int_{-\infty}^{\infty}\int_0^\infty u^sv^{s/2}e^{-u(Xv+Y)}\frac{du}{u}\frac{dv}{v}=\frac{1}{4}\int_{-\infty}^{\infty}\int_0^\infty u^{s/2}v^{s/2}e^{-(Xv+Yu)}\frac{du}{u}\frac{dv}{v}.$$

Finally, change v to v/X and u to u/Y to get

$$\int_{-\infty}^{\infty}\frac{dt}{(Xe^{2t}+Ye^{-2t})^s}=\frac{\Gamma(s/2)\Gamma(s/2)}{4\Gamma(s)}X^{-s/2}Y^{-s/2}.$$

Specify to the problem at hand,

$$\begin{aligned}\int_{\pm\,\mathrm{id}\backslash H_\alpha} \frac{(\operatorname{Im} h_{\mathfrak{a}} z_\alpha)^s}{|m h_{\mathfrak{a}} z_\alpha + n|^{2s}} &= 2^{-s}\sqrt{D}^s \int_{-\infty}^{\infty} \frac{dt}{(2A(\alpha m+n)^2 e^{2t} + 2A(\overline{\alpha} m + n)^2 e^{-2t})^s} \\ &= \frac{\Gamma(s/2)\Gamma(s/2)}{4\Gamma(s)} (2A\alpha\overline{\alpha} m^2 + 4A(\alpha+\overline{\alpha})mn + 2An^2)^{-s} \\ &= \frac{\Gamma(s/2)\Gamma(s/2)}{4\Gamma(s)} (Cm^2 + Bmn + An^2)^{-s}.\end{aligned}$$

That is, the sum of the integrals is

$$\sum_{\{(m,n)\neq 0\}/H'_\alpha} \int_{\pm\,\mathrm{id}\backslash H_\alpha} \frac{(\operatorname{Im} h_{\mathfrak{a}} z_\alpha)^s}{|m h_{\mathfrak{a}} z_\alpha + n|^{2s}} = \frac{\Gamma(s/2)\Gamma(s/2)}{4\Gamma(s)} \sum_{\{(m,n)\neq 0\}/H'_\alpha} \frac{1}{|Cm^2 + Bmn + An^2|^s}.$$

We recognize the latter as a sum over a quadratic form, which corresponds to a sum over a norm of a class of ideals. The index of the sum can be viewed as integral bases, modulo norm 1 units. Thus, by integrating the Eisenstein series over a suitable closed geodesic, we can obtain the sum over any class of ideals.