HOMOGENEOUS SPACES AS QUOTIENTS OF GROUPS

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This statement and proof draws heavily on the exposition of Paul Garrett, in the appendix of `http://www.math.umn.edu/~garrett/m/mfms/notes/02_solenoids.pdf`.

Let X be a locally compact Hausdorff space and G be a topological group acting continuously, transitively on X. Fix a point $x \in X$ and let G_x be the isotropy subgroup of G at x.

Claim 1. The G-*space* X is homeomorphic to the *quotient space* G/G_x under the assignment

$$gG_x \mapsto gx$$

Proof. By the transitivity of the G-action on X, the map $gG_x \mapsto gx$ surjects. Because G_x fixes x, the map injects. To prove the claim, it suffices to show that the map is continuous and open.

The topology on the quotient with projection $\pi : G \to G/G_x$ is uniquely characterized by the condition that any continuous map out of G that is constant on G_x factors uniquely through π to a continuous map out of G/G_x. The map $g \mapsto gx$ is continuous as a restriction of the action, and is constant on G_x by definition of isotropy. Thus $g \mapsto gx$ factors uniquely through π to a continuous map out of G/G_x. The map $gG_x \mapsto gx$ fits the bill, so is continuous.

To prove that $gG_x \mapsto gx$ is open, let U be a neighborhood of $g \in G$. For reasons that will become apparent later, we want a compact neighborhood V of 1 so that $gV^2 = \{gvh : v, h \in V\} \subset U$. To show such a compact set V exists first show the result at $g = 1$.The inverse image of the open U under the (continuous) product map $h \times k \mapsto hk$ is again open. Open sets in the product topology are generated by products of opens in the producands, so the inverse image of U under multiplication contains a product of opens $W_1 \times W_2$ each containing 1. Let $W = W_1 \cap W_2$ so that $W^2 \subset W_1 \cdot W_2 \subset U$ where the last containment comes from the definition of $W_1 \times W_2$ as a subset of the inverse image of U under multiplication. Furthermore G is Hausdorff, so there is some neighborhood W' of 1 contained in W such that $\overline{W'}$ is compact and sits inside W. Let $V = \overline{W'}$ so that $V^2 \subset W^2 \subset U$. For generic g with neighborhood U, the open $g^{-1}U$ is a neighborhood of 1, and the above discussion gives the result. We can *balance* V about 1 (i.e. make it such that $V = V^{-1}$) by setting $V \mapsto V \cap V^{-1}$.

Next, we show that the whole group G can be covered by countably many translates of the compact V. First, we show the result for some open W in V. Let $\{U_1, U_2, ...\}$ be a (countable) basis for G. For each $g \in G$, by the definition of basis, the open gW is the union of those $U_i \subset gW$. As such, for each $g \in G$ there is a smallest index index $j(g)$ such that $g \in U_{j(g)} \subset gW$. For each index i pick some g_i in $j^{-1}(i)$ so that $g_i \in U_i \subset g_i W$. By definition of the map $g \mapsto j(g)$, we have $j^{-1}(i) \subset U_i \subset g_i W$. Taking the union over all (countably many) indices i, $\cup j^{-1}(i) = G \subset \cup g_i W$ as desired. We can certainly replace W by its compact superset V so that $G = \cup g_i V$ as claimed.

We are now ready to prove that the map $gG_x \mapsto gx$ is open. Recall that U is a neighborhood of some point $g \in G$, V is a balanced compact in U such that $V^2 \subset U$. We want to show that Ux is open. Recall the version of the *Baire category theorem*:

> *A locally compact Hausdorff space is not a countable union of nowhere dense sets*

In particular, by transitivity of the group action we can cover the space X by countably many Vx translates $X = \cup g_i V x$. Note that each translate $g_i V x$ is closed, being the continuous image of a compact $g_i V$ in a Hausdorff space. By Baire, some $g_m V x$ contains a nonempty open S. Let h be such that $g_m h x \in S$ and write

$$gx = g(g_m h)^{-1}(g_m h)x \in gh^{-1}g_m^{-1}S$$

The rightmost set in the above display is again open in X because translation in X by a fixed element of G is a homeomorphism. Compute

$$\begin{aligned} gx \in gh^{-1}g_m^{-1}S &\subset gh^{-1}g_m^{-1}g_m V x && \text{(By definition of S)} \\ &\subset gh^{-1}Vx \\ &\subset gV^{-1}Vx \\ &= gV^2 x && (V \text{ is balanced about } 1) \\ &\subset Ux && (\text{By definition of } V)\,, \end{aligned}$$

meaning gx is an interior point of Ux. The group element $g \in U$ was arbitrary so Ux is open, proving the claim. ☐

Remark 1. If[1] X is a smooth manifold and G is a Lie group acting on X smoothly, then the homeomorphism in the conclusion of the above claim is actually a diffeomorphism. Indeed as the isotropy subgroup G_x is closed, the quotient G/G_x has a unique smooth structure so that any smooth map out of G constant on G_x factors uniquely through the projection π to a smooth map out of G/G_x. Because G/G_x is already homeomorphic to G/G_x and (by the mapping property of quotients) the map $f : gG_x \mapsto gx$ is smooth, (by the inverse function theorem) it suffices to show that the differential $df_{1G_x} : T(G/G_x)_{1G_x} \to TX_x$ is nonsingular. Note that the map $h : G \to X$ defined by $g \mapsto gx$ is the composition $f \circ \pi$. Thus, to show that df_{1G_x} is nonsingular, it suffices to show that the kernel of dh_1 is exactly the kernel of $d\pi_1$, i.e. the tangent space $T(G_x)_1$. One direction is easy: $\ker dh_1$ certainly contains $T(G_x)_1$, because h is constant on G_x. To prove the other direction, let $z \in \ker dh_1$ and let Z be the corresponding left invariant vector field on G. Recall that the left invariance of Z is the equality $d(L_\gamma)_{Z(\cdot)} = Z \circ L_\gamma(\cdot)$ where $L_\gamma : g \mapsto \gamma g$ is the (smooth)

[1] I essentially follow Warner in his text *Foundations of Differntiable Manifolds and Lie Groups*, roughly page 120

left action of G on itself. That Z corresponds to z means that Z is the unique vector field such that $\frac{d}{dr}\exp(rZ)|_0 = z$. To show $z \in T(G_x)_1$ it suffices to show that $\exp(tZ) \in G_x$ for all $t \in \mathbb{R}$, meaning $\exp(tZ)$ fixes x for all t. Consider the curve $\alpha : t \mapsto h(\exp(tZ))$ in M. If the tangent vector to α is zero at every t then α is constant. In particular, $\alpha(0) = h(1) = x$ so if α is constant then $\exp(tZ)$ fixes x for all t and is thus in G_x. To prove that the tangent vector to α is zero, first compute for $t = 0$

$$\begin{aligned}
d(\alpha)_0 &= d(h)_1 \circ \frac{d}{dr}\exp(rZ)|_0 \\
&= dh_1(z) && (Z \text{ corresponds to } z) \\
&= 0 && (z \in \ker d(h)_1)\,.
\end{aligned}$$

To prove that $\frac{d}{dr}\alpha(r)|_t = 0$ for all t notice that the map h is invariant under conjugation by a group element γ i.e. $\gamma \cdot h \circ L_\gamma^{-1}(g) = \gamma \cdot \gamma^{-1} \cdot gx = gx = h(g)$. In particular, for $\gamma = \exp(tZ)$ compute

$$\begin{aligned}
\frac{d}{dr}\alpha(r)|_t &= d(h)_{\exp(tZ)} \circ \frac{d}{dr}\exp(rZ)|_t \\
&= d(\exp(tZ) \cdot h \circ L_{\exp(-tZ)})_{\exp(tZ)} \circ \frac{d}{dr}e^{rZ})|_t \\
&= d(\exp(tZ) \cdot h)_{L_{\exp(-tZ)}(\exp(tZ))} \circ \frac{d}{dr}L_{\exp(-tZ)}\exp(rZ)|_t \\
&= d(\exp(tZ) \cdot h)_1 \circ \frac{d}{dr}\exp(rZ)|_0 \\
&= 0
\end{aligned}$$

Thus the curve α is constant, so $\exp(tZ)$ fixes x for all t, meaning the tangent vector z corresponding to Z is in T_1G_x. Therefore $\ker dh_1 = T_1G_x$, so that smooth bijection $f : gG_x \mapsto gx$ has nonsingular derivative, and is thus a diffeomorphism as desired.