MA 261 SPRING 2017: WORKED PROBLEMS

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Problem. 1.11 Given a vector $\vec{v} = (3, 4)$ find two unit vectors that form an angle of $60°$ with $\vec{v}$.

Solution. Call one of the vectors we're looking for $\vec{a} = (a_1, a_2)$. Then the problem tells us that the following equations hold:

$$\vec{a} \cdot \vec{v} = |\vec{v}||\vec{a}| \cos 60° \tag{1}$$

$$|\vec{a}| = 1. \tag{2}$$

In coordinates, using equation (2) to simplify equation (1), and computing $|\vec{v}| = \sqrt{3^2 + 4^2} = 5$, these equations are

$$3a_1 + 4a_2 = 5 \cdot \frac{1}{2} \tag{3}$$

$$a_1^2 + a_2^2 = 1. \tag{4}$$

Solve equation (3) for a_1 in terms of a_2

$$a_1 = \frac{1}{3}(\frac{5}{2} - 4a_2) \tag{5}$$

then plug in to equation (4),

$$\frac{1}{9}(\frac{25}{4} - 2 \cdot \frac{5}{2} \cdot 4a_2 + 16a^2) + a_2^2 = 1$$

which simplifies to

$$-\frac{11}{9} - 20a_2 + 25a_2^2 = 0$$

Solve for a_2 using the quadratic formula:

$$a_2 = \frac{20 \pm \sqrt{20^2 + 25 \cdot 11/9}}{50} = \frac{20 \pm \frac{5\sqrt{155}}{3}}{50}.$$

As discussed in class, there are two solutions, lets give each of them names

$$a_2 = \frac{20 + \frac{5\sqrt{155}}{3}}{50} \quad a_2' = \frac{20 - \frac{5\sqrt{155}}{3}}{50}.$$

Now plugging these values back into equation (5) gives us

$$a_1 = \frac{1}{3}(\frac{5}{2} - 4\frac{20 + \frac{5\sqrt{155}}{3}}{50})$$

$$a_1' = \frac{1}{3}(\frac{5}{2} - 4\frac{20 - \frac{5\sqrt{155}}{3}}{50})$$

Finally, observe that the coordinates of the vectors $\vec{a} = (a_1, a_2)$ and $\vec{a}' = (a_1', a_2')$, both satisfy equations (3) and (4) simultaneously. Thus, $\vec{a}$ and $\vec{a'}$ are two unit vectors which form a 60° angle with $\vec{v}$.