Eigenfunction decompositions of function spaces on various physical domains

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Contents

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Abstract

In the first part of this thesis, we decompose the space of square integrable functions on the torus, the n-sphere, and n-dimensional Euclidean space into eigenspaces of positive, symmetric, unbounded operators. Setting up to use the spectral theorem for compact self-adjoint operators on a Hilbert space, we construct Friedrichs' self adjoint-extension of such operators. We then prove that the resolvent of the extension is compact, provided the domain of the extension embeds compactly into the space of square integrable functions. We prove that the embedding is compact in each of the above spaces, giving an orthonormal basis of eigenfunctions to the resolvent of the extension. To prove that these are genuine eigenfunctions of the original operator we prove various instances of Sobolev regularity, demonstrating that eigenfunctions of the extension lie in domain of the original operator.

In the second part, we apply this technique to decompose the space of square integrable cuspforms on the modular curve $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathfrak{H}$ into eigenspaces of the hyperbolic Laplacian. In contrast to the domains in the first section, there is no known formulaic description of such cuspform eigenfunctions.

For Marisol: you taught me what books couldn't.

Chapter 1

Introducing the circle

A first explanation of Fourier series is that

> A 'nice' function f on the circle is 'represented' by a sum of integer-frequency oscillations $\psi_n(x) = e^{2\pi i n x}$:
>
> $$f \sim \sum_{n\in\mathbb{Z}} c_n \psi_n \quad \text{where } c_n = \int_{\mathbb{T}} f\overline{\psi_n}\,\mathrm{d}\mu$$

For many applications, a mindful treatment of what constitutes 'nice' and what we mean by 'represent' is unnecessary, as most functions used in practice have uniformly convergent Fourier series. Indeed, classical proofs (see [1]) demonstrate that the Fourier series of a function satisfying the 'Dirichlet conditions[1]' converges pointwise to that function, wherever the function is continuous, and to the average of the function values at jump-discontinuities. In contrast to pointwise considerations, a classical proof using Cesaro summation and Fejer kernels shows that $\{\psi_n\}$ forms an orthonormal Hilbert space basis of $L^2(\mathbb{T})$, the square integrable functions (see [29]). In this context there is no notion of pointwise convergence, despite functions satisfying the 'Dirichlet conditions' being a dense subspace.

Even if one needn't discuss convergence properties, one might wonder

> What distinguishes the integer frequency oscillations? Why are they a basis of the 'nice' functions?

I contend that a *proper* answer to this question should suggest a causal mechanism for such decompositions.

In this chapter, we put Fourier analysis into a modern framework: first by designing classes of functions spaces, then by foreshadowing a general mechanism that allows us to both *find* linearly independent orthogonal sets of functions, and *prove* that they span. Developing the theory this way, we position ourselves to do Fourier analysis on spaces for which the bases are not so readily apprent.

---

[1] That is, piecewise differentiable, and left-and-right differentiable at the finitely many points of nondifferentiability

The most immediate issue that arises when analyzing the circle is that it has a distracting amount of structure. For example, the circle is, among other things, an abelian group, a compact topological group, and a one dimensional Lie group. With respect to each of these structures, the integer frequency oscillations $\psi_n(x) = e^{2\pi inx}$ are distinguished by their roles as generators of irreducible representations, continuous characters, and smooth characters. And from each of these interpretations, there is a reasonable proof that the integer frequency oscillations span a relevant space of functions (see [28] for LCA groups, [3] for compact Lie groups via the Peter-Weyl theorem).

With all of this structure, it is not obvious how to adapt our arguments to decompose functions on more general spaces. Should we try to generalize by looking at nonabelian groups? noncompact topological groups? higher dimensional Lie groups?

We can generalize in all three directions simultaneously if we direct our attention towards the *Lie group that acts* on the space[2]: a smooth manifold that is acted on transitively by a Lie group is actually a model of a quotient space of the acting group. A proof of this may be found at [8].

Structures such as measures, topologies, and differential operators that originate in the group, descend to the quotient, and in turn to the space. The acting group need not be abelian, need not be compact, and can have arbitrary (finite) dimension. The following table summarizes various instances of this phenomenon.

The space:	is acted on by:	and is a model of:
the circle	$\mathbb{R}$	$\mathbb{R}/\mathbb{Z}$
the $n-1$ sphere	$\mathrm{O}(n)$	$\mathrm{O}(n)/\mathrm{O}(n-1)$
hyperbolic plane	$\mathrm{SL}_2(\mathbb{R})$	$\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$
hyperbolic 3-space	$\mathrm{SL}_2(\mathbb{C})$	$\mathrm{SL}_2(\mathbb{C})/SU(2)$

Table 1.1: Some smooth manifolds acted on by Lie groups

The circle as a homogeneous space

The relative simplicity of the circle makes it difficult to isolate what drives particular phenomena. In this section we review basic properties of the circle with language that requires little modification to permit a discussion of functions on more general domains.

The Lie group $(\mathbb{R}, +) = \mathbb{R}$ acts transitively on the circle

$$\mathbb{T} = \{e^{2\pi i\theta} : \theta \in \mathbb{R}\}$$

by rotation

$$e^{2\pi i\theta} \times r \mapsto e^{2\pi i(\theta+r)}.$$

The isotropy group in $\mathbb{R}$ of any point on the circle is $\mathbb{Z}$. Thus, as an $\mathbb{R}$-space, the circle is

[2]For a modern treatment of automorphic forms, requiring that group be Lie is too restrictive. In particular, it excludes adelic algebraic groups

a model of the quotient space

$$\begin{aligned} \mathbb{R}/\mathbb{Z} &\approx \mathbb{T} \\ r+\mathbb{Z} &\mapsto e^{2\pi i r}. \end{aligned}$$

For details on the topology of quotient spaces of groups, see [8].

Since $\mathbb{R}$ is abelian, the quotient *space* $\mathbb{R}/\mathbb{Z}$ happens to be a quotient group. For nonabelian groups, isotropy subgroups are generally not normal so the resulting quotient is not a group, but merely a topological space. The isomorphism above connotes that $\mathbb{T}$ and $\mathbb{R}/\mathbb{Z}$ are the same smooth manifold with smooth $\mathbb{R}$ action. In particular, any $\mathbb{Z}$-invariant function on $\mathbb{R}$ can be identified with a function on $\mathbb{T}$ and conversely.

The $\mathbb{R}$-action on $\mathbb{T}$ gives rise to an action on functions. For any $r \in \mathbb{R}$ let R_r be the *right translation action* on any space of functions on $\mathbb{T}$. That is, for $f : \mathbb{T} \to \mathbb{C}$ the function

$$R_r \cdot f : \mathbb{T} \to \mathbb{C}$$

is defined by

$$(R_r \cdot f)(x) = f(x + r).$$

Remark 1. The right translation action induces a *representation* of the Lie group $\mathbb{R}$ on any suitable space of functions on $\mathbb{R}$. Identification of an orthonormal basis through the methods outlined in this chapter is closely related to identifying irreducible subspaces of this representation, in consequence of Schur's lemma. This perspective will not play an active role in this thesis.

Invariant derivative

As a Lie group, $\mathbb{R}$ admits a translation invariant differential operator[3] $\frac{d^2}{dx^2}$.

Translation invariance connotes the equality of operators

$$R_r \circ \frac{d^2}{dx^2} = \frac{d^2}{dx^2} \circ R_r.$$

Being $\mathbb{R}$-translation invariant, $\frac{d^2}{dx^2}$ is certainly $\mathbb{Z}$-invariant so $\frac{d^2}{dx^2}$ is sensible for functions on the quotient $\mathbb{R}/\mathbb{Z}$. Call the image of $\frac{d^2}{dx^2}$ under the isomorphism $\mathbb{R}/\mathbb{Z} \approx \mathbb{T}$ *the Laplacian* Δ.

[3]The reader will note that $\frac{d^2}{dx^2}$ is an order two differential operator, and may wonder why we are not using the invariant operator $\frac{d}{dx}$ instead. The fact that the circle admits a translation invariant first order differential operator is circumstantial. In general the invariant differential operator for functions on a Lie group (or a quotient thereof) comes from the universal enveloping algebra of the group's Lie algebra. That is, the invariant operator typically comes from a bigger space than the Lie algebra, and is typically *quadratic*.

Invariant integral

As a locally compact group, $\mathbb{R}$ admits a canonical translation invariant measure $\mathrm{d}\mu$. Translation invariance means that for an integrable function $f : \mathbb{R} \to \mathbb{C}$ and any $r \in \mathbb{R}$

$$\int_{\mathbb{R}} R_r \cdot f \, \mathrm{d}\mu = \int_{\mathbb{R}} f \, \mathrm{d}\mu.$$

In the present context, we recognize that up to scale, $\mathrm{d}\mu$ is the Lebesgue measure. Since $\mathrm{d}\mu$ is $\mathbb{R}$-translation invariant, it is certainly $\mathbb{Z}$-translation invariant. Now there are two techniques for defining an integral on the quotient $\mathbb{R}/\mathbb{Z}$, and thereby on the circle $\mathbb{T}$.

The first is by integrating on the fundamental domain. A continuous function $f : \mathbb{T} \to \mathbb{C}$ determines a continuous function $F : [0, 1) \to \mathbb{C}$ and we could define

$$\int_{\mathbb{T}} f \, \mathrm{d}\overline{\mu} = \int_0^1 F \, \mathrm{d}\mu.$$

In the present case, there is no harm in doing this, but as we will discuss momentarily, one should exercise restraint in relying on fundamental domains. In particular, even if one can find a fundamental domain, an explicit parameterization may be inaccessible. In that case, explicit computation of the integral may be untenable.

The second is through a characterization. Given continuous function F on $\mathbb{R}$ with compact support, we average F

$$f(x) = \sum_{n \in \mathbb{Z}} F(x + n)$$

to get a $\mathbb{Z}$-invariant function. Given such an F and f, we characterize the integral of f on $\mathbb{T}$ via the integral of F on $\mathbb{R}$ via

$$\int_{\mathbb{T}} f(x) \, \mathrm{d}\overline{\mu}(\overline{x}) = \int_{\mathbb{R}} F(x) \, \mathrm{d}\mu(x).$$

In order for this characterization to be sensible, one must check that every continuous function f can be obtained this way. That is, that every function f on the circle is the average of *some* function on the line. As discussed in [24] this is the case for general reasons. We take for granted that there is an invariant integral on the circle, and we will write the measure as $\mathrm{d}\mu$.

Remark. When endowing $\mathbb{T} \approx \mathbb{R}/\mathbb{Z}$ with structure coming from $\mathbb{R}$, we must be careful with how we think about quotients. One one hand, one can think of the quotient $\mathbb{R}/\mathbb{Z}$ in terms of a *fundamental domain*, say $[0, 1)$. Choosing to think of the quotient in this way, we risk forgetting that the topology and smooth structure are *not* endowed by the ambient Euclidean space. Indeed, a continuous function (on the circle), for example, must agree at 0 and 1. A smooth function must have a left derivatives at 0 and a right derivatives at 2π, and they must agree to every order. The simple nature of the circle makes finding and working on a fundamental domain tenable, but in larger cases it can be highly nontrivial to *find* a fundamental domain, let alone put coordinates on it. Thus, we will try to establish a habit of working on the quotient, where there is no distracting ambient topology.

1.0.1 Functions on $\mathbb{T}$

There are two essentially distinct ways to define function spaces on the circle. Classically, one defines the spaces as collections of complex valued functions satisfying certain analytic conditions, and then equipping it with a suitable definition of convergence. In the modern framework one defines each function space as the metric space completion of smooth functions with respect to certain norm/inner-product/metric.

Remark. The author's intention is not to emphasize a dichotomy between the two descriptions. Rather, the fact that there are two descriptions of the same spaces should be seen as a feature of the theory. Thinking classically can make the spaces in question more tangible, but makes both completeness and the relation of the space to others less clear. Conversely, the modern description incorporates completeness and mapping properties into the design, but may obscure the nature of the elements of the space. We will find that the modern description clarifies the mechanism behind spectral decompositions, thus fits better with our present needs.

As complex vectorspaces, the function spaces on the circle are easily described:

- $L^2(\mathbb{T})$, which quantifies regularity of functions through a notion of 'average size,' and incorporates the structure of $\mathbb{T}$ as a measure space.
- $C^{2k}(\mathbb{T})$ for $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, a family which quantifies regularity of functions in terms of their smoothness.

To make them amenable to analysis, these spaces are topologized so as to incorporate the relevant notion of regularity into the definition of convergence.

Whatever these spaces are, and however they are topologized, it is essential that they fit into the diagram

$$C^\infty(\mathbb{T}) \to \cdots \to C^{2(k+1)}(\mathbb{T}) \to C^{2k}(\mathbb{T}) \to \cdots \to C^0(\mathbb{T}) \to L^2(\mathbb{T})$$

where an arrow connotes continuous inclusion, and further that each included image is dense. We want the included images to be dense because it allows us specify continuous maps on larger spaces such as $L^2(\mathbb{T})$ or $C^0(\mathbb{T})$ by specifying them on smaller spaces sitting to the left.

In chapter 2, we will discover a family of *Hilbert spaces* $H^k(\mathbb{T})$ continuously interweaving between the *Banach spaces* $C^{2k}(\mathbb{T})$, the former being a better setting for analysis.

Function spaces: classical

For a complete discussion of the classical set-up, see [27]

$C^k(\mathbb{T})$: For each nonnegative integer k, define the complex vectorspace

$$C^{2k}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} : (\frac{d^2}{dx^2})^k f \text{ is continuous}\}.$$

We understand that $(\frac{d^2}{dx^2})^0$ is the identity, making $C^0(\mathbb{R})$ the space of continuous functions on $\mathbb{R}$. Set

$$C^\infty(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} : \frac{d^2}{dx^2} \text{ can be applied to } f \text{ indefinitely}\}.$$

We say a sequence of functions $f_n \in C^{2k}(\mathbb{R})$ converges to $f \in C^{2k}(\mathbb{R})$ if

$$(\frac{d^2}{dx^2})^m f_n \to (\frac{d^2}{dx^2})^m f$$

uniformly for each $m \leq 2k$.

We say a sequence of smooth functions $f_n \in C^\infty(\mathbb{R})$ converges to $f \in C^\infty(\mathbb{R})$ if

$$(\frac{d^2}{dx^2})^m f_n \to (\frac{d^2}{dx^2})^m f$$

uniformly for every m.

Define the linear subspace $C^{2k}(\mathbb{T})$ of right $\mathbb{Z}$ invariant functions in $C^{2k}(\mathbb{R})$, and let Δ be the restriction of $\frac{d^2}{dx^2}$ to such functions. The description of sequence convergence above descends to the quotient, giving a definition of $C^{2k}(\mathbb{T})$ convergence.

From the description of convergence above, it is apparent that Δ is continuous as a map $C^{2(k+1)}(\mathbb{T}) \to C^{2k}(\mathbb{T})$, and as an operator $C^\infty(\mathbb{T}) \to C^\infty(\mathbb{T})$.

$L^2(\mathbb{T})$: With our invariant integral defined on $\mathbb{T}$, we now introduce the space $L^2(\mathbb{T})$. Provisionally define it as a space of functions, try

$$\{f : \mathbb{T} \to \mathbb{C} : f \text{ measurable, and } \int_{\mathbb{T}} |f|^2 \, \mathrm{d}\mu < \infty\}.$$

One would like to define convergence of a sequence $f_n \in L^2(\mathbb{T})$ to $f \in L^2(\mathbb{T})$ by

$$f_n \to f \iff \int_{\mathbb{T}} |f - f_n|^2 d\mu \to 0.$$

However, because integration is not sensitive to function values on sets of measure zero, this notion of convergence does not specify limits uniquely. Remedy this by defining

$$L^2(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{C} : f \text{ measurable, and } \int_{\mathbb{T}} |f|^2 \, \mathrm{d}\mu < \infty\}/\sim$$

where $f \sim g$ connotes agreement off a set of measure zero.

The diagram By definition, convergence of a sequence in $C^{2(k+1)}(\mathbb{T})$ is a stronger condition than convergence in $C^{2k}(\mathbb{T})$ so the inclusion

$$C^{2(k+1)}(\mathbb{T}) \to C^{2k}(\mathbb{T})$$

is continuous. For the same reason, the inclusion

$$C^{\infty}(\mathbb{T}) \to C^{2k}(\mathbb{T})$$

is continuous for every k. Since the circle is compact, every continuous function is square integrable. Thus we can identify $C^0(\mathbb{T})$ with a subspace in $L^2(\mathbb{T})$. Moreover if a sequence of continuous functions converges uniformly, then it certainly converges on average, making

$$C^0(\mathbb{T}) \to L^2(\mathbb{T})$$

continuous.

Combining these facts, we do indeed find

$$C^{\infty}(\mathbb{T}) \to \cdots \to C^{2(k+1)}(\mathbb{T}) \to C^{2k}(\mathbb{T}) \to \cdots \to C^0(\mathbb{T}) \to L^2(\mathbb{T})$$

However, the fact that each subspace is dense is not immediate.

Remark. As discussed in the next section, the spaces $C^{2k}(\mathbb{T})$ topologized above can be given a compatible norm, making them Banach, and $L^2(\mathbb{T})$ can be given an inner product to make it Hilbert. However, $C^{\infty}(\mathbb{T})$ cannot be given a norm to make it Banach. Instead, it is Frechét, with its topology given by a countable family of seminorms. The description of the topology in terms of limit convergence above does not make it clear that the topologies come from a norm or inner product, much less that the spaces are complete with respect to such.

Function spaces: modern

For details on the following descriptions, see [11].

$C^{\infty}(\mathbb{T})$: Starting just as an untopologized infinite dimensional complex vectorspace, define

$$C^{\infty}(\mathbb{T}) = \mathbb{Z}\text{-invariant functions in } C^{\infty}(\mathbb{R})$$

where as before, we define $C^{\infty}(\mathbb{R})$ to be those functions on which $\frac{d^2}{dx^2}$ can be applied indefinitely. Again, Δ is the restriction of $\frac{d^2}{dx^2}$ to $C^{\infty}(\mathbb{T})$. Once we have defined the spaces $C^{2k}(\mathbb{T})$ we will return to $C^{\infty}(\mathbb{T})$ to topologize it as a limit.

$C^k(\mathbb{T})$: For each $k \in \mathbb{Z}_{\geq 0}$ define the norm on $C^{\infty}(\mathbb{T})$

$$|\cdot|_{C^{2k}(\mathbb{T})} : f \longmapsto \sum_{m \leq k} \left(\sup_{x \in \mathbb{T}} |\Delta^m f(x)| \right).$$

And define

$$C^{2k}(\mathbb{T}) = \text{ completion of } C^\infty(\mathbb{T}) \text{ with respect to } |\cdot|_{C^{2k}(\mathbb{T})}.$$

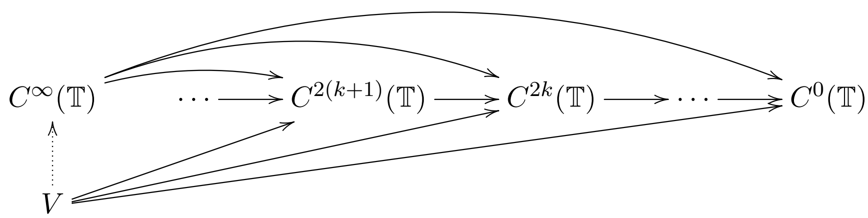
By design, each $C^{2k}(\mathbb{T})$ is complete. Since $|\cdot|_{\mathbb{C}^{2k}(\mathbb{T})}$ dominates $|\cdot|_{\mathbb{C}^{(2(k-1)}(\mathbb{T})}$ there is a continuous injection from the former to the latter, obtained by identifying $C^\infty(\mathbb{T})$ limit points. This reestablishes the chain of continuous inclusions

$$\cdots \to C^{2(k+1)}(\mathbb{T}) \to C^{2k}(\mathbb{T}) \to C^{2(k-1)}(\mathbb{T}) \to \cdots \to C^0(\mathbb{T}),$$

Now, being totally explicit about the topology of the source, we have the family of isometric injections

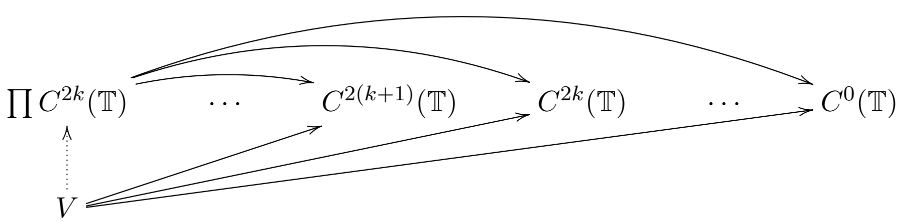
$$(C^\infty(\mathbb{T}), |\cdot|_{C^{2k}(\mathbb{T})}) \to C^{2k}(\mathbb{T}).$$

In considering this family of maps, we notice that the topology of the source varies from map to map. What we want is a single topology on $C^\infty(\mathbb{T})$ that makes each inclusion into each $C^{2k}(\mathbb{T})$ continuous. Further we stipulate that this is the coarsest such topology, in the sense that for *any* suitable object[4] V mapping continuously to each $C^{2k}(\mathbb{T})$ compatibly with the transitions maps, there is a unique continuous map $V \to C^\infty(\mathbb{T})$ through which the compatible maps factor. That is, there exists a unique map $V \to C^\infty(\mathbb{T})$ making the following diagram commute:



This description characterizes such a topology uniquely, but that there actually *is* such a topology requires a construction, as follows.

The characterization of of the topology on $C^\infty(\mathbb{T})$, immediately above, is a strengthening of the characterization of the product. Indeed, recall that the product $\prod C^{2k}(\mathbb{T})$ is the coarsest topological vectorspace continuously mapping to each $C^{2k}(\mathbb{T})$. That is, for any topological vectorspace V, any collection of continuous maps $V \to C^{2k}(\mathbb{T})$ will factor through $\prod C^{2k}(\mathbb{T})$. Diagrammatically,



One notes the similarity with this digram to that immediately above, with the only difference being the absence of horizontal transition maps between the $C^{2k}(\mathbb{T})$. Ignoring topology for

[4] In the present circumstance, a discussion of what auxiliary objects should be used in this characterization may be distracting. At the very least, the auxiliary object should be a complex vectorspace and should carry some topology that makes the vectorspace operations continuous. However, these constraints are too weak for a workable category. A further technical assumption of local convexity suffices to discuss products, coproducts, limits and colimits. Fortunately, every function space that we discuss in this thesis is locally convex.

a moment, letting $V = C^\infty(\mathbb{T})$ and the maps $V \to C^{2k}(\mathbb{T})$ be inclusions, we see that there is a unique set inclusion $C^\infty(\mathbb{T}) \to \prod C^{2k}(\mathbb{T})$. Thus, we may identify $C^\infty(\mathbb{T})$ with a subset of $\prod C^{2k}(\mathbb{T})$ consisting of (left sided) sequences of $C^\infty(\mathbb{T})$ diagonally embedded:

$$f \mapsto (\dots, f, f, f) \in \dots \times C^4(\mathbb{T}) \times C^2(\mathbb{T}) \times C^0(\mathbb{T}).$$

The product $\prod C^{2k}(\mathbb{T})$ has a unique topology, equipping the image of $C^\infty(\mathbb{T})$ with the subspace topology[5].

With $C^\infty(\mathbb{T})$ topologized, we can now show that it satisfies the mapping property above. Any auxiliary object V and collection of maps $V \to C^{2k}(\mathbb{T})$ compatible with the inclusions $C^{2k}(\mathbb{T}) \to C^{2(k-1)}(\mathbb{T})$ uniquely maps to the product, as discussed above. Thus, there is a unique candidate map $V \to C^\infty(\mathbb{T})$ making the *product* diagram commute. The compatibility of the maps $V \to C^{2k}(\mathbb{T})$ with the inclusions is exactly the condition for this map to make the *limit* diagram commute.

As desired, we have set up the chain of continuous inclusions

$$C^\infty(\mathbb{T}) \to \dots \to C^{2(k+1)}(\mathbb{T}) \to C^{2k}(\mathbb{T}) \to \dots \to C^0(\mathbb{T})$$

and the included image of each space is dense, by design. Further, for finite k, the spaces $C^{2k}(\mathbb{T})$ are Banach, again by design.

$L^2(\mathbb{T})$: Define the inner product on $C^\infty(\mathbb{T})$

$$\langle \cdot, \cdot \rangle_{L^2(\mathbb{T})} : f \times g \longmapsto \int_{\mathbb{T}} f\overline{g}\, \mathrm{d}\mu$$

and let $|\cdot|_{L^2(\mathbb{T})}$ be the resulting norm. Now define

$$L^2(\mathbb{T}) = \text{ completion of } C^\infty(\mathbb{T}) \text{ with respect to } |\cdot|_{L^2}(\mathbb{T}).$$

Again, by design, $L^2(\mathbb{T})$ is complete and thus a Hilbert space and $(C^\infty(\mathbb{T}), |\cdot|_{L^2(\mathbb{T})})$ is a dense subspace. Since the circle is compact, each of the $C^{2k}(\mathbb{T})$ norms dominate the $L^2(\mathbb{T})$ norm in $C^\infty(\mathbb{T})$ so we have a new bottom for the chain of continuous inclusions

$$C^\infty(\mathbb{T}) \to \dots \to C^{2(k+1)}(\mathbb{T}) \to C^{2k}(\mathbb{T}) \to \dots \to C^0(\mathbb{T}) \to L^2(\mathbb{T}).$$

[5]Though a discussion of the topology on distributions (that is, continuous linear functionals on $C^\infty(\mathbb{T})$) will not play a leading role in this thesis, one should nonetheless note that the diagonally embedded copy of $C^\infty(\mathbb{T})$ in $\prod C^{2k}(\mathbb{T})$ (with the subspace topology) is closed, being the intersection of sets of the form

$$\{(\dots, f_2, f_0) \in \prod C^{2k}(\mathbb{T}) : \mathrm{inc}_{2k}\, f_{2k} = f_{2(k-1)} \text{in } C^{2(k-1)}(\mathbb{T})\}$$

where $\mathrm{inc}_{2k} : C^{2k}(\mathbb{T}) \to C^{2(k-1)}(\mathbb{T})$ is inclusion. Since the inclusion maps are continuous, each of these sets is closed. Just as $C^\infty(\mathbb{T})$ can be identified with a *closed subspace* of *the product*, the distributions $C^\infty(\mathbb{T})^*$ can be identified with a *quotient* of *the coproduct* by a *closed subspace*. The resulting topology on distributions is tamed by the closedness of the subspace.

1.0.2 Integer frequency oscillations, revisted

With the definitions of $L^2(\mathbb{T})$ and $C^\infty(\mathbb{T})$ established, we are ready to reinterpret the distinguishing property of the integer frequency oscillations. As discussed above, translation invariant differential operator $\frac{d^2}{dx^2}$ acts on $C^\infty(\mathbb{R})$

$$\frac{d^2}{dx^2} : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}).$$

For nonzero λ, the differential equation

$$\frac{d^2}{dx^2} f = \lambda f$$

has the a family of smooth solutions

$$\psi_z(x) = e^{2\pi i z x} \qquad \text{when } \lambda = z \in \mathbb{C}^\times.$$

When $\lambda = 0$ the corresponding solutions are $x \mapsto x$ and $x \mapsto 1$.

Among these solutions, only those which are $\mathbb{Z}$ invariant descend to the circle $\mathbb{T} \approx \mathbb{R}/\mathbb{Z}$. One checks

$$\psi_z(x+1) = e^{2\pi i z(x+1)} = e^{2\pi i z} e^{2\pi i z} = e^{2\pi i z} \psi_z(x)$$

showing that the necessary and sufficient condition for $\mathbb{Z}$ invariance is that $z \in \mathbb{Z}$.

On the circle, the translation invariant differential operator Δ acts on smooth functions

$$\Delta : C^\infty(\mathbb{T}) \to C^\infty(\mathbb{T})$$

and from above, the differential equation

$$\Delta f = \lambda f$$

has the discrete family of solutions

$$\psi_n(x) = e^{2\pi i n x} \qquad \text{when } \lambda = n \in \mathbb{Z}.$$

That is, the integer frequency oscillations are the eigenfunctions of the translation invariant differential operator Δ on $C^\infty(\mathbb{T})$. The fact that the $\psi_n(x)$ form an orthonormal basis of $L^2(\mathbb{T})$ can be reformulated as

Theorem 1. $L^2(\mathbb{T})$ has an orthonormal basis of eigenfunctions for the translation invariant differential operator Δ.

As mentioned earlier, a smooth manifold X acted on transitively by a Lie group G with isotropy subgroup[6] G_x of a point $x \in X$ is a model of the quotient G/G_x. A Lie group G admits a G-translation invariant differential operator

$$\Omega : C^\infty(G) \to C^\infty(G),$$

[6]That is, the subgroup of G that fixes x.

which descends to the quotient as an operator

$$\Delta^X : C^\infty(X) \to C^\infty(X)$$

by restricting to right G_x invariant functions.

Also, G admits a translation invariant measure $\mathrm{d}\mu$ which descends to the quotient $G/G_x \approx X$, provided the modular function of G restricts to that of G_x. Define a norm on $C^\infty(X)$

$$|f|^2_{L^2(X)} = \int_X |f|^2 \, \mathrm{d}\mu.$$

Then, define

$$L^2(X) = \text{ completion of } C^\infty(X) \text{ with respect to } |\cdot|_{L^2(X)}.$$

Now the theorem above is amenable to a generalization

Question. Does $L^2(X)$ have an orthonormal basis of eigenfunctions for Δ^X?

The objective of this thesis is not to treat the question in complete generality, in part because the answer is sometimes *no*. Indeed, letting $X = \mathbb{R}$ with $G = \mathbb{R}$ acting on X, we see that $G_0 = \{0\}$ giving the obvious $X \approx G$. Then the invariant operator is $\Delta^{\mathbb{R}} = \frac{d^2}{dx^2}$ and the invariant measure is the Lebesgue measure. The differential equation

$$\frac{d^2}{dx^2} f = \lambda f$$

has the solutions $\psi_z(x) = e^{2\pi i z x}$ as above. However, *none* of these solutions are square integrable! There are *no* eigenfunctions for $\Delta^{\mathbb{R}}$ in $L^2(\mathbb{R})$, and in particular, there can be no basis of such.

Instead, the objective of this thesis is to demonstrate particular smooth manifolds X and groups G affirming the question above, and to give a general framework for proving such a decomposition.

1.0.3 Towards a causal mechanism

As outlined in the first section, the integer frequency oscillations play important roles depending on how we view the circle: they are continuous or smooth characters and irreducible representations of the compact abelian Lie group $\mathbb{T}$. Without viewing $\mathbb{T}$ as a group, but rather as an $\mathbb{R}$-space with translation invariant Δ and $\mathrm{d}\mu$, the property that distinguishes them is that they are the smooth eigenfunctions of Δ, discussed in the last section. Acknowledging this fact, we should shift our attention to studying the properties of the operator Δ, rather than the functions themselves. In doing so, we hope to make Fourier series be a consequence of general Hilbert space theory.

Using integration by parts twice, compute for smooth f and g,

$$\langle \Delta f, g \rangle = \int_{\mathbb{T}} \frac{d^2}{dx^2} f \overline{g} \, \mathrm{d}\mu = \int_{\mathbb{T}} f \overline{\frac{d^2}{dx^2}(g)} = \langle f, \Delta g \rangle.$$

Thus, Δ is a *symmetric* operator on $C^\infty(\mathbb{T})$ in $L^2(\mathbb{T})$. That is, on $C^\infty(\mathbb{T})$, Δ and its adjoint agree. However, the domain of the adjoint of Δ is properly larger than $C^\infty(\mathbb{T})$.

Recall from linear algebra that any *self-adjoint* operator on a finite dimensional complex vectorspace is unitarily diagonalizable. That is, there is an orthonormal basis of eigenvectors for that operator. The infinite dimensional extension of this theorem says that a *compact, self-adjoint* operator on a Hilbert space has an orthonormal basis of eigenvectors. The operator Δ is not self-adjoint, as mentioned. Further it cannot be compact, because it is not continuous, even on polynomials. Indeed, working on the fundamental domain $[0, 2\pi)$, compute the L^2 norm of $f_n(x) = (x/2\pi)^n$ is $1/(n+1)$ but the norm of $\Delta f_n(x)$ is $(n-1)n/(n+1)$, showing that $\lim \Delta f_n \neq \Delta \lim f_n$.

Yet, $L^2(\mathbb{T})$ *does* have an orthonormal basis of eigenfunctions of Δ. Further, this phenomenon is not specific to the circle! For example:

The eigenfunctions of the spherical Laplacian are an orthonormal basis for $L^2(S^{n-1})$. That is, for $f \in L^2(S^{n-1})$

$$L^2(S) = \bigoplus_{d\geq 0} \mathfrak{H}_d$$

where $\mathfrak{H}^d$ is an orthonormal basis of harmonic polynomials homogeneous of degree d in $\mathbb{C}[x_1, .., x_n]$ restricted to the sphere.

The square integrable eigenfunctions of the Schrödinger operator $r^2 - \Delta^{\mathbb{R}^n}$ are a basis of $L^2(\mathbb{R}^n)$,

$$L^2(\mathbb{R}^n) = \bigoplus_{d\geq 0} \left(\bigoplus_{a=0}^{\lceil d/2 \rceil} R^a(\mathfrak{H}^{d-2a} \cdot g) \right)$$

where $g(x) = e^{-|x|/2}$ is the Gaussian, and R is the *raising operator*, to be defined later. Note that $r^2 - \Delta^{\mathbb{R}^n}$ is not the $\mathbb{R}$ translation invariant operator coming from the universal enveloping algebra, which we demonstrated does *not* have any square integrable eigenfunctions.

The cuspform eigenfunctions of the hyperbolic Laplacian are an orthonormal basis of all square integrable cuspforms $L^2(\Gamma\backslash\mathfrak{H})_{\text{cfm}}$ Notably, this is the case without any formulaic description of the cuspform eigenfunctions.

The operators mentioned in each of these decompositions are *symmetric* (*not* self-adjoint), unbounded (*not* compact), and merely densely defined. Yet, their eigenfunctions do form an orthonormal basis. Somehow operators for which the spectral theorem can *almost* be applied, are still producing spectral theorem-like conclusions. One mechanism underlying these phenomena is Friedrichs's construction of a self-adjoint extension to positive symmetric densely defined operators. A feature of this construction is that under certain circumstances, the resolvent of the self-adjoint extension is *compact*. Further, in certain cases, Levi-Sobolev theory shows that the process of extending the operator does not introduce new eigenfunctions. Thus, granting that these assertions are true, we will be able to honestly apply the spectral theorem for compact self adjoint operators to arrive at the decompositions above.

Chapter 2

Unbounded Operators

In the first chapter, we identified the integer frequency oscillations on the circle as a family of eigenfunctions of the densely defined unbounded symmetric operator Δ. Despite the integer frequency oscillations forming an orthonormal Hilbert space basis for $L^2(\mathbb{T})$, the spectral theorem does not apply to such operators. In this chapter, we construct the self-adjoint Friedrichs extension for positive densely defined symmetric operators. In doing so, we will discover a Sobolev space as the natural domain such an extension.

In the process of constructing this extension we will find that under certain conditions, the resolvent of the self-adjoint extension will be compact, thus giving an orthonormal basis of eigenvectors to the resolvent. In the next chapter, we will use Levi-Sobolev theory on specific Friedrichs extensions to show that in some cases, the process of extending the operator does not introduce new eigenvectors. That is the eigenvectors of the resolvent of the extension are eigenvectors of the original.

This chapter draws heavily on [19] and [27], though I have taken some liberties in reorganizing Friedrichs' construction.

2.1 Functional analytic preliminaries

One of the principal reasons that Hilbert spaces are a better working environment is that their duals are easy to describe. This is made precise via the *Riesz–Fischer theorem*: given a bounded linear functional $\lambda : V \to \mathbb{C}$ on a Hilbert space V, there is a unique vector $w_\lambda = w \in V$ so that

$$\lambda(v) = \langle v, w\rangle \quad \text{for all } v \in V.$$

Given a bounded linear map $T : (V_1, \langle\cdot,\cdot\rangle_1) \to (V_2, \langle\cdot,\cdot\rangle_2)$ and a vector $w \in V_2$, a natural linear functional to consider is

$$\lambda_T(v) = \langle Tv, w\rangle_2.$$

This functional is bounded, by Cauchy–Schwarz, so Riesz–Fischer assures us that there is some vector $w' \in V_1$ so that this linear functional is given by the inner product

$$\lambda_T(v) = \langle v, w'\rangle_1.$$

We can apply Riesz–Fischer in this way for any choice of $w \in V_2$, and we get a verifiably bounded linear association of vectors which we define as the *adjoint*

$$T^* : (V_2, \langle \cdot, \cdot \rangle_2) \to (V_1, \langle \cdot, \cdot \rangle_1) \quad \text{via} \quad w \mapsto w'$$

While the construction of an adjoint to a bounded map between Hilbert spaces above *does* generalize to bounded maps between pre-Hilbert spaces (that is, possibly incomplete inner product spaces), the argument above *fails* to define an adjoint for an *unbounded* operator on a dense subspace of a Hilbert space, such as Δ on $C^\infty(\mathbb{T}) \subset L^2(\mathbb{T})$. In particular the linear functional

$$\lambda_\Delta(v) = \langle \Delta v, w \rangle$$

need not be bounded for all $w \in L^2(\mathbb{T})$, so Riesz–Fischer needn't apply.

Nonetheless, we observed that if $g \in C^\infty(\mathbb{T})$, the linear functional on $C^\infty(\mathbb{T})$

$$\lambda_\Delta(f) = \langle \Delta f, g \rangle_{L^2}$$

is given by

$$\lambda_\Delta(f) = \langle f, \Delta g \rangle_{L^2}$$

by integrating by parts twice. However, if $g \in L^2(\mathbb{T})$ is in $C^2(\mathbb{T})$ but not in any smaller $C^{2k}(\mathbb{T})$, the integration by parts is formally sensible, but we may not conclude that $g'' = \Delta g$ because g is not in the domain of Δ, as defined.

Thus, evidently, with this choice of domain for Δ, there are operators Δ' with bigger domains D' than Δ that fit into the adjunction

$$\langle \Delta f, g \rangle = \langle f, \Delta' g \rangle \quad \text{for } f \in C^\infty(\mathbb{T}) \text{ and } g \in D'.$$

Conceivably, there could be some operator Δ' with domain D' fitting into the adjunction that behaves radically different from Δ on $D' - C^\infty(\mathbb{T})$ even if Δ seems like it could be defined there.

In order to work with unbounded operators like Δ, apparently the choice of *domain* is an essential part of the description of the operator. In what follows, we will only consider unbounded[1] operators with *dense domain*. Explicitly, define an **unbounded operator** on a Hilbert space V to be a pair (T, D_T) with dense $D_T \subset V$, and $T : D_T \to V$.

The collection of unbounded operators admits a partial ordering $(T, D_T) \subset (S, D_S)$ if $D_T \subset D_S$ and $S|_{D_T} = T$. Of course, unbounded operators need not be comparable in general. When the domain is clear from context, we may write an unbounded operator as just T, and the partial ordering as $T \subset S$ and we say S extends T.

Two examples of unbounded operators are $(\Delta, C^\infty(\mathbb{T}))$ and $(\Delta, C^2(\mathbb{T}))$, and in this case $(\Delta, C^\infty(\mathbb{T})) \subset (\Delta, C^2(\mathbb{T}))$.

Abstracting from the discussion of adjoints and Δ above, for an unbounded operator (T, D_T) a **subadjoint** to T is an unbounded operator $(T', D_{T'})$ fitting into the adjunction

$$\langle Tv, w \rangle = \langle v, T'w \rangle \quad \text{for } v \in D_T \text{ and } w \in D_{T'}\ .$$

[1]Unbounded connotes that the operator is not necessarily bounded

As with $(\Delta, C^\infty(\mathbb{T}))$, an unbounded operator that is a subadjoint to itself is called **symmetric**. Notice that every subadjoint $(T', D_{T'})$ to a symmetric unbounded operator (T, D_T) naturally extends that operator by $(T', D_{T'} + D_T)$.

Integrating by parts twice, as we did above, shows that $(\Delta, C^2(\mathbb{T}))$ is a subadjoint to to the symmetric operator $(\Delta, C^\infty(\mathbb{T}))$.

For a fixed unbounded operator (T, D_T) let $(T_1', D_{T'})$ and $(T_2', D_{T'})$ be two subadjoints with common domain,

$$\langle Tv, w\rangle = \langle v, T_1'w\rangle = \langle v, T_2'w\rangle \quad \text{for } v \in D_T \text{ and } w \in D_{T'}.$$

Then compute

$$0 = \langle v, (T_1' - T_2')w\rangle.$$

Because D_T is dense this shows that $(T_1', D_{T'}) = (T_2', D_{T'})$. In particular this means that the partial order on the subadjoints of a fixed unbounded operator reduces to a partial order on containment of domains.

Next, we show that a symmetric unbounded operator, such as $(\Delta, C^\infty(\mathbb{T}))$ admits a unique subadjoint with maximal domain. To this end, introduce an isometry of $V \oplus V$,

$$u : v \oplus w \longmapsto -w \oplus v.$$

First observe that $u^2 = -1$, then compute

$$\langle u(v \oplus w), x \oplus y\rangle = \langle -w, x\rangle + \langle v, y\rangle = \langle v \oplus w, y \oplus -x\rangle = \langle v \oplus w, -u(x \oplus y)\rangle.$$

Consequently, u commutes with orthogonal complement formation: if t is orthogonal to uX then $-ut$ is orthogonal to X. That is, $u(X^\perp) = (uX)^\perp$, so we can unambiguously write $uX^\perp$.

Claim 1. A symmetric operator (T, D_T) has a unique maximal subadjoint (T^*, D_{T^*}). We call this maximal subadjoint *the adjoint* of T. Further, T^* is closed, in the sense that it has a closed graph.

Proof. Subadjointness can be characterized as an orthogonality condition: let $(T', D_{T'})$ be any subadjoint to T, $v \in D_T$ and $w \in D_{T'}$. Then the relation

$$\langle Tv, w\rangle = \langle v, T'w\rangle$$

rephrases as

$$\langle w \oplus T'w, u(v \oplus Tv)\rangle = 0.$$

Universalizing over $v \in D_T$ and $w \in D_{T'}$, we have

$$\langle \operatorname{graph} T', u \operatorname{graph} T\rangle = 0,$$

which is exactly to say

$$\operatorname{graph} T' \subset u \operatorname{graph} T^\perp.$$

If $u \operatorname{graph} T^\perp$ is a graph, then the function it defines will be the maximal subadjoint to T. To check that it is a graph, fix $w \in H$ and suppose there are two $v, v' \in H$ such that

$w \oplus v \in u\,\mathrm{graph}\,T^{\perp}$ and $w \oplus v' \in u\,\mathrm{graph}\,T^{\perp}$. Recall that $u(w \oplus v) = -v \oplus w$. Then we have both $w \oplus v$ and $w \oplus v'$ are orthogonal $-Tx \oplus x$ for all $x \in D_T$. Subtracting the two candidates shows that $0 \oplus (v - v')$ is orthogonal to all of $0 \oplus D_T$. Further D_T is dense in H, so $v = v'$.

Thus, for any $w \in H$ there is at most one v such that $w \oplus v \in u\,\mathrm{graph}\,T^{\perp}$. The assignment $T^* : w \mapsto v$ is linear because $u\,\mathrm{graph}\,T^{\perp}$ is a vectorspace, and is maximal among all subadjoints, because the graph of any subadjoint is a subspace of the graph of T^*. Thus, T^* is characterized by

$$\mathrm{graph}\,T^* = u\,\mathrm{graph}\,T^{\perp}.$$

As an orthogonal complement, T^* has a closed graph. □

The characterization of the adjoint to a symmetric unbounded operator in terms of orthogonality to a graph grants several immediate corollaries (assuming T to be symmetric and densely defined):

- Since orthogonal complementation reverses inclusion, so does adjoint formation $T_1 \subset T_2 \implies T_2^* \subset T_1^*$.
- Since T is a subadjoint to T^*, the double-adjoint extends T: $T \subset T^{**}$.

The first bullet formalizes one's intuition that a smaller domain induces a weaker constraint on (and thus a larger domain for) the adjoint. The second bullet demonstrates that every symmetric operator has a closed extension, though that extension need not be self-adjoint.

Recall that a symmetric unbounded operator is a subadjoint to itself, but it need not be *maximal*. A stronger condition is **self-adjointness**, whereby a symmetric unbounded operator is *equal* to its adjoint, the maximal subadjoint. Because adjoints have closed graphs, the graph of a self-adjoint operator is closed.

One of the central principles of finite dimensional spectral theory is that a self-adjoint operator has real spectrum. For unbounded operators on a (infinite dimensional) Hilbert space, we can relax the requirement of self-adjointness:

Claim 2. Eigenvalues for symmetric operators are real

Proof. Let T be a symmetric operator defined on D, and let $v \in D$ be nonzero, with $Tv = \lambda v$. Compute

$$\lambda\langle v, v\rangle = \langle \lambda v, v\rangle = \langle Tv, v\rangle = \langle v, Tv\rangle = \langle v, \lambda v\rangle = \bar{\lambda}\langle v, v\rangle$$

because T agrees with its adjoint on D. Thus λ is real. □

In finite dimensional linear algebra, with an operator T acting on a vectorspace V and $\lambda \in \mathbb{C}$, the conditions

- $T - \lambda$ is not injective, meaning λ is an eigenvector of T

- $T - \lambda$ is not surjective, but has dense image
- $T - \lambda$ is not surjective, and does not have dense image
- $T - \lambda$ has a densely defined inverse[2]

are equivalent, by dimension counting. However, on infinite dimensional vectorspaces, these conditions are demonstrably inequivalent. Define the spectrum $\sigma(T)$ of an unbounded operator T to be the set of $\lambda \in \mathbb{C}$ for which any of the above bullets fail. In particular, the spectrum of an operator contains more than just eigenvalues. With this in mind, we define

Definition 1. The resolvent of T at $\lambda \in \mathbb{C}$ is the operator $R_\lambda = (T - \lambda)^{-1}$, whenever it is continuous and densely defined.

As a continuous operator, the resolvent has a better spectral theory than the not necessarily bounded operator from which it is defined. The following theorem shows that beyond just the eigenvalues, the whole spectrum of a self-adjoint operator is real.

Theorem 2. Let (T, D_T) be self-adjoint with dense domain D_T. For all non real $\lambda \in \mathbb{C}$

- $R_\lambda = (T - \lambda)^{-1}$ is a resolvent
- R_λ is defined on all of V.

If, in addition, T is positive then the same result holds as long as λ is not nonnegative real.

Proof. First we show that $T - \lambda$ injects for any $\lambda = x + iy \notin \mathbb{R}$, making R_λ well defined on the image. Compute

$$\langle (T - \lambda)v, (T - \lambda)v \rangle = |(T - x)v|^2 + iy\langle (T - x)v, v \rangle - iy\langle v, (T - x)v \rangle + y^2|v|^2.$$

Because T is symmetric and x is real, $T - x$ is symmetric on D_T. Then the cross terms in the preceding display cancel,

$$|(T - \lambda)v|^2 = |(T - x)v|^2 + y^2|v|^2 \geq y^2|v|^2. \tag{2.1}$$

We have chosen λ to be nonreal, so y^2 is positive. For nonzero v, $(T - \lambda)v$ is nonzero, so $T - \lambda$ injects.

Now we show $T - \lambda$ surjects, so that the resolvent $R_\lambda = (T - \lambda)^{-1}$ is defined everywhere. First show that the image is dense. Let w be orthogonal to $(T - \lambda)D_T$. If $(T - \lambda)^*$ is defined on w, then $(T - \lambda)^*w$ is orthogonal to D_T by construction. The latter is dense, so $(T - \lambda)^*w = 0$. Further, $(T - \lambda)^*w = 0$ is compatible with the definition of the adjoint because for all $v \in D_T$

$$\langle (T - \lambda)v, w \rangle = 0 = \langle v, 0 \rangle.$$

[2]On a finite dimensional vectorspaces all linear maps are continuous, so there is no need to discuss densely defined maps. For infinite dimensional vectorspaces, it is possible that an injective map does not have dense image, and thus the inverse would not be densely defined, and thus would not be an operator in our current framework.

Thus, $w \in D_{T^*}$ and $T^*w = \bar{\lambda}w$. Because T is self-adjoint, $w \in D_T$, and $Tw = \lambda w$. By the last claim, all of the eigenvalues of a self-adjoint operator are *real*. Thus $w = 0$, meaning $T - \lambda$ has dense image.

To show that $T - \lambda$ surjects, it now suffices to show that $(T - \lambda)D_T$ is closed. From the corollary above, T has closed graph in $V \oplus V$. Trivially, the scalar operator λ has closed graph, so graph$(T - \lambda)$ is closed.

One would like to conclude that the continuous image of graph$(T - \lambda)$ under projection is closed, but continuous maps need not preserve closedness. However, from the bound at the beginning of the proof, we can say more about this projection than mere continuity. Namely it *respects metrics*. The essential point is that metric respecting maps preserve *completeness*.

Define the projection map on the graph of $T - \lambda$,

$$F : v \oplus (T - \lambda)v \longmapsto (T - \lambda)v.$$

From (2.1), for all $v \in D_T$

$$|v|^2 \leq \frac{1}{y^2}|(T - \lambda)v|^2$$

where y^2 is positive and fixed. On the other hand, trivially

$$|(T - \lambda)v|^2 \leq |(T - \lambda)v|^2 + |v|^2.$$

Combining, we have

$$|(T - \lambda)v|^2 \leq |(T - \lambda)v|^2 + |v|^2 \leq (1 + \frac{1}{y^2})|(T - \lambda)v|^2$$

Which is to say for any $z \in \operatorname{graph} T$,

$$|F(z)|^2 \leq |z|^2 \leq (1 + \frac{1}{y^2})|F(z)|^2,$$

so F respects metrics.

By the first claim, the self-adjoint operator T has a closed graph in $V \oplus V$, so $T - \lambda$ has a closed graph. A closed subspace in a complete space is itself complete, so graph$(T - \lambda)$ is complete. Then F preserves completeness, so $F(\text{graph}(T - \lambda)) = (T - \lambda)D$ is complete in V. A complete metric subspace is closed, because convergent sequences are always Cauchy in a metric space. So in addition to being dense, $(T - \lambda)D_T$ is closed in V. Thus $(T - \lambda)D_T = V$ meaning $T - \lambda$ surjects its domain onto V. Therefore, $(T - \lambda)^{-1} = R_\lambda$ is everywhere defined.

To see that R_λ is continuous, let $v = R_\lambda w$ and use the lower bound on $T - \lambda$

$$|(T - \lambda)v| \geq y|v|,$$

as an upper bound on the operator norm of R_λ

$$|R_\lambda w| \leq \frac{1}{y}|w|,$$

proving boundedness, thereby continuity.

When T is positive, in addition to being self-adjoint, we can relax our conditions on λ, as $\mathrm{Re}(\lambda) < 0$. Then we get a similar lower bound on the norm of $T - \lambda$

$$\begin{aligned} |(T-\lambda)v|^2 &= |Tv|^2 - \lambda\langle Tv, v\rangle - \bar{\lambda}\langle v, Tv\rangle + |\lambda v|^2 \\ &= |Tv|^2 - 2\,\mathrm{Re}(\lambda)\langle Tv, v\rangle + |\lambda v|^2 \end{aligned}$$

by observing $\mathrm{Re}(\lambda) < 0$, and T is positive, making the cross term positive so

$$|(T-\lambda)v|^2 = |Tv|^2 + 2|\Re(\lambda)|\langle Tv, v\rangle + |\lambda v|^2 \geq |\lambda|^2|v|^2.$$

This estimate allows the proof to repeat verbatim. □

In the next section we will find that Friedrichs' construction of a self-adjoint extension to a positive, symmetric, unbounded operator proceeds through a characterization of a resolvent.

2.2 Friedrichs' construction of a self-adjoint extension

As discussed in the first section, a discussion of unbounded operators and their adjoints requires attention to the choice of domain. In particular, we found that specifying a larger domain for a symmetric operator results in a smaller domain for its adjoint. On the other hand, the adjoint to a symmetric operator extends that operator. Apparently there are genuine constraints in play when finding the domain of an extension of a symmetric unbounded operator.

The following theorem, due originally to K.O. Friedrichs, constructs such an extension. The *construction* is useful, because it provides auxiliary information about the extension, whereas comparable axiom of choice arguments merely assert existence.

A salient feature of the construction is that it provides a glimpse of a proto-Sobolev space as the natural domain of the extension.

Theorem 3. Let (T, D_T) be a positive, symmetric, densely defined operator on a Hilbert space $(V, \langle\cdot,\cdot\rangle)$. There exists a positive self-adjoint extension $(\widetilde{T}, D_{\widetilde{T}})$ of (T, D_T). Further, the resolvent $(1+\widetilde{T})^{-1}$ is characterized by the condition

$$\langle (1+T)v, (1+\widetilde{T})^{-1}w\rangle = \langle v, w\rangle \qquad (\text{for all } v \in D_T \text{ and } w \in V)$$

Proof. Define the new Hermitian form on D_T.

$$\langle v, w\rangle_1 = \langle (1+T)v, w\rangle \quad \text{for } v, w \in D_T$$

Since T is positive, the new form is positive, and because $1+T$ injects, the form is definite, so $(D_T, \langle\cdot,\cdot\rangle_1)$ is a pre-Hilbert space. Despite $(D_T, \langle\cdot,\cdot\rangle_1)$ and $(D_T, \langle\cdot,\cdot\rangle)$ having the same underlying set, they are not necessarily the same Hilbert spaces. Nonetheless, there is a linear *set* bijection

$$\iota : (D_T, \langle\cdot,\cdot\rangle_1) \to (D_T, \langle\cdot,\cdot\rangle).$$

The positivity of T shows that for $v \in (D_T, \langle\cdot,\cdot\rangle_1)$, the new norm dominates the original

$$|\iota v|^2 \le |v|_1^2 = \langle (1+T)\iota v, \iota v\rangle,$$

making the inclusion $\iota : (D_T, \langle\cdot,\cdot\rangle_1) \to (D_T, \langle\cdot,\cdot\rangle)$ continuous.

Let V_1 be the Hilbert space completion of $(D_T, \langle\cdot,\cdot\rangle_1)$, and recall that V is the Hilbert space completion of $(D_T, \langle\cdot,\cdot\rangle)$. The continuous inclusion

$$\iota : (D_T, \langle\cdot,\cdot\rangle_1) \to (D_T, \langle\cdot,\cdot\rangle)$$

extends to a continuous inclusion of completions, still called

$$\iota : (V_1, \langle\cdot,\cdot\rangle_1) \to (V, \langle\cdot,\cdot\rangle).$$

Because $(D_T, \langle\cdot,\cdot\rangle_1)$ is dense in $(V_1, \langle\cdot,\cdot\rangle_1)$, we may view $1+T$ as a densely defined unbounded operator

$$(1+T)' : (V_1, \langle\cdot,\cdot\rangle_1) \to (V, \langle\cdot,\cdot\rangle),$$

and $(1+T)' = (1+T)\iota$ as maps $(V_1, \langle\cdot,\cdot\rangle_1) \to (V, \langle\cdot,\cdot\rangle)$. For the remainder, we suppress the superscript, but still replace $(1+T)\iota$ when context demands.

A continuous map between Hilbert spaces has a continuous adjoint $\iota^* : (V, \langle\cdot,\cdot\rangle) \to (V_1, \langle\cdot,\cdot\rangle_1)$, which satisfies

$$\langle \iota v, w\rangle = \langle v, \iota^* w\rangle_1 \quad \text{(for all } v \in V_1 \text{ and } w \in V).$$

In particular, if $v \in D_T$, then we can think of $\iota(v)$ as v and after expanding the definition of $\langle\cdot,\cdot\rangle_1$ the adjunction above becomes

$$\langle v, w\rangle = \langle (1+T)v, \iota^* w\rangle \quad \text{(for all } v \in D_T \text{ and } w \in V).$$

We recognize this to be the asserted characterization of $(1+\widetilde{T})^{-1} : V \to V$ in the statement of the theorem. Thus we anticipate that its inverse, so long as it exists, will be the extension $(1+\widetilde{T})$.

We are concerned with operators on $(V, \langle\cdot,\cdot\rangle)$ rather than on the auxiliary space $(V_1, \langle\cdot,\cdot\rangle_1)$, we follow the adjoint $\iota^* : V \to V_1$ with the inclusion $\iota : V_1 \to V$ to define the operator $\iota\iota^* : V \to V$. Now we establish some properties of $\iota\iota^*$

- $\iota\iota^*$ is *bounded*: being the composition of a continuous inclusion and its continuous adjoint.

- $\iota\iota^*$ is *self-adjoint*: since $(\iota\iota^*)^* = \iota^{**}\iota^* = \iota\iota^*$.

- $\iota\iota^*$ is *positive*: compute for any $v \in V$ that $\langle \iota\iota^* v, v\rangle = \langle \iota^* v, \iota^* v\rangle_1 \ge 0$, using the definition of adjoint.

- $\iota\iota^*$ is *injective*: if $\iota\iota^* w = 0$, then $0 = \langle \iota\iota^* w, w\rangle = \langle \iota^* w, \iota^* w\rangle_1$ making $w = 0$.
- $\iota\iota^*$ has $\langle\cdot,\cdot\rangle$-*dense image*: if $\langle \iota\iota^* v, w\rangle = 0$ for all $v \in V$, then in particular for $v = w$, we have $0 = \langle \iota\iota^* w, w\rangle = \langle \iota^* w, \iota^* w\rangle_1$ making $w = 0$.

Note that ι^* has image in $(V_1, \langle\cdot,\cdot\rangle_1)$, so we may see the image of $\iota\iota^*$ as being in the copy of V_1 in V.

Thus $\iota\iota^*$ has a positive, symmetric, densely defined, potentially unbounded inverse $A : (D_A, \langle\cdot,\cdot\rangle) \to (V, \langle\cdot,\cdot\rangle)$ where $D_A = \iota\iota^* V$. We anticipate that (A, D_A) is the self-adjoint extension $(1 + \widetilde{T}, D_{\widetilde{T}})$ of $(1 + T, D_T)$. To complete the proof it suffices to show that A is self adjoint, that A extends $1 + T$, and that $\langle Av, v\rangle \geq \langle v, v\rangle$ so that $\widetilde{T}$ is positive.

To show that A is self-adjoint, introduce the two operators on $V \oplus V$

$$\begin{aligned} u &: v \oplus w \longmapsto -w \oplus v \\ s &: v \oplus w \longmapsto w \oplus v. \end{aligned}$$

Recall from the first section that u commutes with orthogonal complement formation, that $u^2 = -1$, and that the graph of the adjoint to an unbounded operator L is characterized by

$$\operatorname{graph} L^* = u \operatorname{graph} L^\perp.$$

Observe that for an invertible operator L,

$$\operatorname{graph} L^{-1} = s \operatorname{graph} L,$$

that $usX = suX$ (setwise) for any subspace X, and that s also commutes with orthogonal complement formation. Now compute

$$\begin{aligned} \operatorname{graph} A^* &= u \operatorname{graph} A^\perp && \text{Characterization of adjoint} \\ &= us \operatorname{graph} \iota\iota^{*\perp} && \iota\iota^* \text{ inverts } A \\ &= su \operatorname{graph} \iota\iota^{*\perp} && su = us \text{ setwise on a subspace} \\ &= s \operatorname{graph} \iota\iota^* && \iota\iota^* \text{ is self-adjoint} \\ &= \operatorname{graph} A && A \text{ inverts } \iota\iota^*, \end{aligned}$$

proving that A is self-adjoint.

To show that A extends $1 + T$, compute for $w \in (D_T, \langle\cdot,\cdot\rangle_1)$ and $w' = \iota\iota^* v \in D_A$

$$\langle (1 + T)w, w'\rangle = \langle (1 + T)w, \iota\iota^* v\rangle = \langle w, v\rangle = \langle w, Aw'\rangle$$

using the characterization of $\iota\iota^* = (1 + \widetilde{T})^{-1}$ in the statement of the theorem. Thus the self-adjoint operator A is a subadjoint to the symmetric operator $1 + T$, and thereby an extension.

To show that $\langle Av, v\rangle \geq \langle v, v\rangle$, take $v = \iota\iota^* w$ and compute

$$\langle Av, v\rangle = \langle w, \iota\iota^* w\rangle = \langle \iota^* w, \iota^* w\rangle_1 \geq \langle \iota\iota^* w, \iota\iota^* w\rangle = \langle v, v\rangle$$

Thus, the operator $\widetilde{T} = A - 1$ is positive.

Thus, with $D_{T^*} = \iota\iota^* V$ and $A = 1 + \widetilde{T}$, the densely defined unbounded operator

$$1 + \widetilde{T} : (D_{T^*}, \langle \cdot, \cdot \rangle) \to (V, \langle \cdot, \cdot \rangle)$$

is a positive, self-adjoint extension of $1 + T$ satisfying

$$\langle v, w \rangle = \langle (1+T)v, (1+\widetilde{T})^{-1} w \rangle \quad \text{For all } v \in D_T \text{ and } w \in V.$$

□

In the case that the inclusion $(V_1, \langle \cdot, \cdot \rangle_1)$ is *compact*, the fact that the resolvent $(1+\widetilde{T})^{-1}$ to the Friedrichs extension $\widetilde{T}$ arises as the composition of the continuous adjoint $\iota^* : (V, \langle \cdot, \cdot \rangle) \to (V_1, \langle \cdot, \cdot \rangle_1)$ followed by the *compact* inclusion $\iota : (V_1, \langle \cdot, \cdot \rangle_1) \to (V, \langle \cdot, \cdot \rangle)$ allows us to say more,

Corollary 1. *If the inclusion $\iota : (V_1, \langle \cdot, \cdot \rangle_1) \to (V, \langle \cdot, \cdot \rangle)$ is* compact *then the resolvent $(1 + \widetilde{T})^1 = \iota\iota^*$ is* compact.

Proof. The resolvent $(1+\widetilde{T})^{-1}$ is the composition of a continuous map ι^* with the compact map ι. To show that the composition is compact, scale ι^* to have operator norm 1 so that the image of the unit ball in $(V, \langle \cdot, \cdot \rangle)$ under ι^* is a subset of the unit ball in $(V_1, \langle \cdot, \cdot \rangle_1)$. Since ι is compact, the image of the unit ball in $(V_1, \langle \cdot, \cdot \rangle_1)$ under ι has compact closure. A closed subset of compact is compact, so the image of the unit ball in $(V, \langle \cdot, \cdot \rangle)$ under $\iota\iota^*$ has compact closure in $(V, \langle \cdot, \cdot \rangle)$, as claimed. □

Chapter 3

Finding Bases

3.0.1 Rehearsing the meta-argument

Last chapter, we constructed Friedrichs' *self-adjoint* extension $\widetilde{T} : D_{\widetilde{T}} \to V$ to a positive, symmetric, unbounded operator $T : D_T \to V$. The domain $D_{\widetilde{T}}$ of the extension is a dense subspace of a Sobolev-like Hilbert space

$$V_1 = \text{completion of } D_T \text{ with respect to } |v|_1^2 = |v|^2 + \langle Tv, v \rangle.$$

The construction focused on the *resolvent* $(1+\widetilde{T})^{-1} : V \to V_1$, a continuous, everywhere defined, self-adjoint map that satisfies the adjoint-like condition

$$\langle v, w \rangle = \langle (1+T)v, (1+\widetilde{T})^{-1} w \rangle \quad \text{for all } v \in D_T \text{ and } w \in V.$$

The inclusion $V_1 \to V$ is always continuous, by the positivity of T. If the inclusion is *compact*, then $(1+\widetilde{T})^{-1} : V \to V$ is also compact, being the composition of a continuous map with the compact inclusion.

Thus, when $V_1 \to V$ is compact, the *spectral theorem* for compact self-adjoint operators asserts that V has an orthonormal Hilbert space basis of eigenvectors of the *resolvent* of the extension. For purely algebraic reasons, the spectrum of $1+\widetilde{T}$ and the nonzero spectrum of $(1+\widetilde{T})^{-1}$ are in bijection. Indeed, first recall that the spectral theorem for compact self-adjoint operators assures us that the spectrum of $(1+\widetilde{T})^{-1}$ is all point spectrum i.e., eigenvalues. The formal identities

$$(1+\widetilde{T})^{-1} - \lambda^{-1} = (1+\widetilde{T})^{-1}(\lambda - (1+\widetilde{T}))\lambda^{-1}$$

and

$$(1+\widetilde{T}) - \lambda = (1+\widetilde{T})(\lambda^{-1} - (1+\widetilde{T})^{-1})\lambda$$

show that the failure of $(1+\widetilde{T})^{-1} - \lambda^{-1}$ to be invertible forces that of $(1+\widetilde{T}) - \lambda$, and conversely. Consequently the nonzero (point) spectrum of $1+\widetilde{T}$ and $(1+\widetilde{T})^{-1}$ are identical under the bijection $\lambda \mapsto \lambda^{-1}$. That $1+\widetilde{T}$ is invertible ensures that 0 is not an eigenvalue.

In order to prove that the basis consists of eigenvectors for T itself, we must prove that the process of extension does not introduce new eigenvectors. Generally, to prove that the eigenvectors of $\widetilde{T}$ are eigenvectors of T itself, we introduce a pertinent notion of **Sobolev regularity**: roughly speaking, eigenvectors of $\widetilde{T}$ must lie in the original domain of T, whereat $\widetilde{T}$ and T agree.

In this chapter, we apply this argument to find orthonormal bases for functions on the torus $\mathbb{T}$, the $n-1$ sphere S^{n-1}, and n dimensional Euclidean space $\mathbb{R}^n$.

We postpone the spectral decomposition of $L^2(\Gamma\backslash\mathfrak{H})$ to the next chapter because of the substantial dependence on as yet undisscused terminology.

My exposition on the decompositions of $L^2(\mathbb{T})$ and $L^2(\mathbb{R}^n)$ draw from [19], though the treatment of $\mathbb{R}^n$ rather than $\mathbb{R}$ is my own work. For the decomposition of $L^2(S^{n-1})$ I worked from [17], but the intrinsic description of the operators is my own slight amplification. My exposition on the manifestation of the oscillator representation, after the decomposition of $L^2(S^{n-1})$, recounts my discovery of that structure in this context—though I learned about the representation from [22].

3.1 Discrete decomposition of $L^2(\mathbb{T})$

Take $C^\infty(\mathbb{T}) = D_\Delta$ as the domain of the unbounded operator $\Delta = \frac{d^2}{dx^2}$. Integrating by parts, citing that $\mathbb{T}$ has no boundary, shows that $-\Delta$ is *positive semi-definite*

$$\langle \Delta f, f\rangle = \int_{\mathbb{T}} f'' f \,\mathrm{d}\mu = -\int_{\mathbb{T}} f' f' \,\mathrm{d}\mu = -|f|^2_{L^2(\mathbb{T})} \quad \text{for } f \in C^\infty(\mathbb{T}) \ .$$

Integration by parts twice shows that Δ is *symmetric*

$$\langle \Delta f, g\rangle = \langle f, \Delta g\rangle \quad \text{for } f, g \in C^\infty(\mathbb{T}).$$

Thus, $-\Delta$ has a positive self-adjoint extension $-\widetilde{\Delta}$ characterized in terms of its resolvent,

$$\langle f, g\rangle = \langle (1-\Delta)f, (1-\widetilde{\Delta})^{-1} g\rangle \quad \text{for } f \in C^\infty(\mathbb{T}) \text{ and } g \in L^2(\mathbb{T}).$$

The domain of the extension $\widetilde{\Delta}$ is a dense subspace of $H^1(\mathbb{T})$, the +1-Sobolev space

$$H^1(\mathbb{T}) = \text{completion of } C^\infty(\mathbb{T}) \text{ with respect to } |f|_1^2 = |f|^2_{L^2} + |f'|^2_{L^2}$$

Compactness of the inclusion $H^1(\mathbb{T}) \longrightarrow L^2(\mathbb{T})$

From the last chapter, to prove that the resolvent $(1-\widetilde{\Delta})^{-1}$ is compact, it suffices to prove that the inclusion $H^1(\mathbb{T}) \to L^2(\mathbb{T})$ is compact.

Claim 3. (Rellich) The inclusion $H^1(\mathbb{T}) \to L^2(\mathbb{T})$ is compact

Proof. First, observe that the $H^1(\mathbb{T})$ norm dominates the $L^2(\mathbb{T})$ norm on $C^\infty(\mathbb{T})$,

$$|f|^2_{H^1(\mathbb{T})} = \langle (1-\Delta)f, f\rangle = |f|^2_{L^2(\mathbb{T})} + |f'|^2_{L^2(\mathbb{T})} \geq |f|^2_{L^2(\mathbb{T})}.$$

Thus, there is a continuous injection $H^1(\mathbb{T}) \to L^2(\mathbb{T})$.

Compactness of the inclusion $H^1(\mathbb{T}) \to L^2(\mathbb{T})$ connotes that the the included image of a compact set in $H^1(\mathbb{T})$ has compact closure in $L^2(\mathbb{T})$. In a complete metric space, having compact closure is equivalent to being totally bounded. That is, given any $\varepsilon > 0$, the image can be covered by finitely many ε balls. Further, since any compact set in $H^1(\mathbb{T})$ is bounded, we may assume it is contained in the unit $H^1(\mathbb{T})$-ball. Thus, it suffices to prove that the set $\{f \in C^\infty(\mathbb{T}) : |f|_{H^1(\mathbb{T})} \leq 1\}$ is totally bounded in $L^2(\mathbb{T})$. The claim follows after extending by continuity.

First, we compute a pointwise bound for smooth f satisfying $|f|_{H^1(\mathbb{T})} \leq 1$, by showing such functions satisfy a *Lipschitz* condition. Indeed, for such f, working with coordinates $x, y \in [0,1]$ and using the fundamental theorem of calculus and then Cauchy–Schwarz–Bunyakovski,

$$|f(x) - f(y)| = |\int_y^x (f' \cdot 1)| \leq |f'|_{L^2(\mathbb{T})}|x-y|^{1/2} \leq |f|_{H^1(\mathbb{T})}|x-y|^{1/2} \leq |x-y|^{1/2}.$$

Pairing this computation with the definition of the $+1$ Sobolev norm, $|f|^2_{H^1(\mathbb{T})} = |f|^2_{L^2(\mathbb{T})} + |f'|^2_{L^2(\mathbb{T})} \leq 1$ so we may assume that $|f(x)| \leq 2$ for all $x \in [0,1]$.

Fix $\varepsilon > 0$, and choose $\delta > 0$ and $n \in \mathbb{Z}_{\geq 0}$ subordinate to ε, satisfying $n^{1/2}(n^{-1/2} + \delta) < \varepsilon$. Cover the radius 2 disk in $\mathbb{C}$ with N δ-balls, $\{B(c_j, \delta) : j = 1, ..., N\}$. For any list $J = (j_1, ..., j_n)$ with each $j_i \in \{1, .., N\}$, define the simple function

$$F_J : [0,1] \to \mathbb{C} \quad \text{via} \quad F_j(x) = c_i \quad \text{for } x \in [i/n, (i+1)/n).$$

Note that there are N^n such functions, a finite number.

Now we show that the N^n balls $B(F_J, \varepsilon)$ in $L^2(\mathbb{T})$ cover $\{f \in L^2(\mathbb{T}) : |f|_{H^1(\mathbb{T})} \leq 1\}$. For any such f, we use our cover of the the radius 2 disk in $\mathbb{C}$ to find a list of c_{j_i} such that $|f(i/n) - c_{j_i}| < \delta$ for each $i \in \{1, ..., n\}$. From this collection, form the list $J = (j_1, ..., j_n)$, and define the corresponding function F_J. Now compute for $x \in [i/n, (i+1)/n)$, using the Lipschitz condition for the second inequality

$$\begin{aligned} |f(x) - F_J(x)| &= |f(x) - c_{j_i}| \\ &\leq |f(x) - f(i/n)| + |f(i/n) - c_{j_i}| \\ &< |x - i/n|^{1/2} + \delta \leq n^{-1/2} + \delta. \end{aligned}$$

Compile these n pointwise estimates for the $L^2(\mathbb{T})$ bound

$$|f - F_J|^2_{L^2(\mathbb{T})} = \int_{[0,1]} |f - F_J|^2 \leq n(1/\sqrt{n} + \delta)^2 < \varepsilon^2.$$

This shows $f \in B(F_J, \varepsilon)$, proving the (finite) collection of such balls cover the image of $H^1(\mathbb{T})$ in $L^2(\mathbb{T})$, showing the latter is totally bounded. It follows that the inclusion $H^1(\mathbb{T}) \longrightarrow L^2(\mathbb{T})$ is compact. □

Thus $(1-\widetilde{\Delta})^{-1}$ is compact and self-adjoint, so applying the *spectral theorem* yields an orthonormal basis of $L^2(\mathbb{T})$ consisting of eigenfunctions for $(1-\widetilde{\Delta})^{-1}$. Since the (point) spectrum of $(1-\widetilde{\Delta})^{-1}$ and $1-\widetilde{\Delta}$ are in bijection, these are eigenfunctions of $1-\widetilde{\Delta}$, in turn eigenfunctions of $\widetilde{\Delta}$ itself.

Sobolev regularity

To prove that the process of extending Δ does not introduce new eigenfunctions, it suffices to prove that any eigenfunction of $\widetilde{\Delta}$ is *smooth*, since $\widetilde{\Delta}$ and Δ agree on $C^\infty(\mathbb{T})$. Granting this, the eigenfunctions of Δ comprise an orthonormal basis of $L^2(\mathbb{T})$.

Recall that Friedrichs' construction from chapter 2 showed that the domain of the extension $\widetilde{\Delta}$ is contained in $H^1(\mathbb{T})$, so that $(1-\widetilde{\Delta})^{-1}L^2(\mathbb{T}) \subset H^1(\mathbb{T})$. The eigenfunction condition

$$\widetilde{\Delta} f = \lambda f \quad \text{for } f \in H^1(\mathbb{T})$$

yields $(1-\widetilde{\Delta})f = (1-\lambda)f$, which in turn shows

$$f = (1-\widetilde{\Delta})^{-1}(1-\lambda)f \subset (1-\widetilde{\Delta})^{-1}H^1(\mathbb{T}).$$

In particular, for such an f,

$$\langle (1-\widetilde{\Delta})f, f\rangle_1 = \langle (1-\widetilde{\Delta})(1-\Delta)f, f\rangle < \infty.$$

This computation leads us to discover the family of Hilbert spaces

$$H^k(\mathbb{T}) = \text{completion of } C^\infty(\mathbb{T}) \text{ with respect to } |f|_{H^k(\mathbb{T})} = \langle (1-\Delta)^k f, f\rangle^{1/2},$$

where $k \in \mathbb{Z}_{\geq 0}$. By construction, $(1-\widetilde{\Delta})^{-k}L^2(\mathbb{T}) \subset H^k(\mathbb{T})$. In particular, recapitulating the last computation, we've found that eigenfunctions f for $\widetilde{\Delta}$ satisfy

$$f = (1-\widetilde{\Delta})^{-1}(1-\lambda)f \in H^2(\mathbb{T}).$$

Repeating the computation for every k, we find

$$f \in H^\infty(\mathbb{T}) = \bigcap_{k\in\mathbb{Z}_{\geq 0}} H^k(\mathbb{T}).$$

Recall from chapter 1 that we defined the family of Banach spaces

$$C^k(\mathbb{T}) = \text{completion of } C^\infty(\mathbb{T}) \text{ with respect to } |f|_{C^k(\mathbb{T})} = \sup_{0\leq i\leq k} \sup_{x\in\mathbb{T}} |f^{(i)}(x)|$$

and

$$C^\infty(\mathbb{T}) = \bigcap_{k\in\mathbb{Z}_{\geq 0}} C^k(\mathbb{T}).$$

Since $C^\infty(\mathbb{T})$ is dense in every $H^k(\mathbb{T})$ and $C^k(\mathbb{T})$, the norm domination $|f|_{H^k(\mathbb{T})} \leq |f|_{C^k(\mathbb{T})}$ on $C^\infty(\mathbb{T})$ shows that $C^k(\mathbb{T}) \subset H^k(\mathbb{T})$. Thus, at the very least, $C^\infty(\mathbb{T}) \subset H^\infty(\mathbb{T})$.

Now, to prove that the eigenfunctions of $\widetilde{\Delta}$ are smooth, it suffices to prove that

$$H^\infty(\mathbb{T}) = C^\infty(\mathbb{T}).$$

The *Sobolev* embedding theorem, proved next, asserts the norm dominances

$$c|f|_{H^k(\mathbb{T})} \le |f|_{C^k(\mathbb{T})} \le C|f|_{H^{k+1}(\mathbb{T})}, \qquad c, C > 0 \text{ independent of } f \in C^\infty(\mathbb{T})$$

which implies continuity of the inclusions

$$H^{k+1}(\mathbb{T}) \to C^k(\mathbb{T}) \to H^k(\mathbb{T}).$$

Granting this, we find

$$H^\infty(\mathbb{T}) = \bigcap H^k(\mathbb{T}) = \bigcap H^{k+1}(\mathbb{T}) \subset \bigcap C^k(\mathbb{T}) = C^\infty(\mathbb{T}) \subset H^\infty(\mathbb{T}),$$

demonstrating equality.

Claim 4. (Sobolev) As normed spaces, $H^{k+1}(\mathbb{T}) \subset C^k(\mathbb{T})$.

Proof. First, we establish $H^1(\mathbb{T}) \subset C^0(\mathbb{T})$ by showing that $|f|_{H^1(\mathbb{T})} \le c|f|_{L^2(\mathbb{T})}$ for $f \in C^\infty(\mathbb{T})$ and some $c > 0$. Recall the pointwise bound from the proof of claim 3,

$$\begin{aligned} |f(y) - f(x)| &= \left|\int_x^y f'(t)\,\mathrm{d}t\right| \\ &\le \int_x^y (|f'(t)| \cdot 1)\,\mathrm{d}t \\ &\le |f'|_{L^2(\mathbb{T})}|y - x|^{1/2} \le |f'|_{C^2(\mathbb{T})}. \end{aligned}$$

Now pick $y \in [0, 1]$ such that $|f|$ attains its minimum at y. Further, note that $|f(y)| \le |\int_0^1 f(t)\,\mathrm{d}t| \le \int_0^1 |f(t)|\,\mathrm{d}t \le |f|_{L^2(\mathbb{T})}$. Now compute

$$\begin{aligned} |f(x)| &\le |f(y)| + |f(x) - f(y)| \\ &\le |f|_{L^2(\mathbb{T})} + |f'|_{L^2(\mathbb{T})} \\ &\le 2(|f|^2_{L^2(\mathbb{T})} + |f'|^2_{L^2(\mathbb{T})})^{1/2} = 2|f|_{H^1(\mathbb{T})}. \end{aligned}$$

Since this estimate is independent of x, it shows that the $+1$ Sobolev norm dominates the sup-norm on $C^\infty(\mathbb{T})$, giving the containment of the completions.

Now suppose that the Sobolev $+k$ norm dominates the $k-1$ sup norm on $C^\infty(\mathbb{T})$. Then compute

$$\begin{aligned} |f|_{C^k(\mathbb{T})} &= \sum_{i=0}^{k} \sup_{x\in\mathbb{T}} |f^{(i)}(x)| \\ &\le 2|f|_{H^k(\mathbb{T})} + \sup_{x\in\mathbb{T}} |f^{(k)}(x)| \\ &\le 2|f|_{H^k(\mathbb{T})} + 2|f^{(k)}|_{H^1(\mathbb{T})} \\ &\le 2|f|_{H^k(\mathbb{T})} + 2|f|_{H^{k+1}(\mathbb{T})} \\ &\le 4|f|_{H^{k+1}(\mathbb{T})}, \end{aligned}$$

and the result follows by induction. □

Complete determination of eigenfunctions

We know that the collection of eigenfunctions of Δ comprises an orthonormal basis of $L^2(\mathbb{T})$. In this section, we determine these eigenfunctions

$$\Delta f = \lambda f \quad \text{for } f \in C^\infty(\mathbb{T}) \text{ and } \lambda \in \mathbb{R}_{\leq 0}\ .$$

Put the coordinates $[0, 2\pi]$ on $\mathbb{T}$, so that the equation above becomes the ordinary differential equation with boundary conditions

$$\frac{d^2 f}{dx^2} = \lambda f \quad \text{and} \quad f^{(j)}(0) = f^{(j)}(2\pi) \quad \text{for } \lambda \in \mathbb{R}_{\leq 0},\ j \in \mathbb{Z}_{\geq 0}.$$

When $\lambda \neq 0$, the solutions to the ODE are linear combinations of $e^{\pm\sqrt{\lambda}x}$. The boundary condtion $f(0) = f(2\pi)$ requires that $\sqrt{\lambda} \in i\mathbb{Z} - \{0\}$. So the $\lambda = -n^2 \neq 0$ eigenspace is spanned by $e^{\pm nix}$. When $\lambda = 0$, the solutions to the ODE are linear combinations of the constant function 1 and x. The boundary condition eliminates the latter, so the $\lambda = 0$ eigenspace is spanned by 1.

Conclusion

We have shown

Letting ψ_n denote the integer frequency oscillation on $\mathbb{T}$,

$$L^2(\mathbb{T}) = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} \cdot \psi_n \quad f = \sum_{n \in \mathbb{Z}} \langle f, \psi_n \rangle \psi_n.$$

3.2 Discrete decomposition of $L^2(S^{n-1})$

3.2.1 The sphere as homogeneous space

The sphere $S = S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ is acted on by the special orthogonal group

$$\begin{aligned} \mathrm{SO}(n) &= n \times n \text{ matrices with orthonormal rows, and determinant } 1 \\ &= n \times n \text{ matrices } g \text{ satisfying } gg^\top = 1\text{, with determinant } 1 \\ &= n \times n \text{ matrices that preserve the inner product, and have determinant } 1. \end{aligned}$$

Recall that a smooth manifold X that is acted on transitively by a Lie group G is naturally diffeomorphic to the quotient space G/G_x, where G_x is the isotropy subgroup of any point $x \in X$.

Consider $\mathbb{R}^n$ to be *row* vectors, and let SO(n) act on the right, by traditional matrix-by-vector mutliplication.

The action of SO(n) on S is transitive. Fix any point $v \in S$, we want to construct an element $g \in \mathrm{SO}(n)$ that takes the nth unit vector e_n to v. Take the bottom row of g to be v. Use Gram–Schmidt to obtain an orthonormal basis $x_1, ..., x_{n-1}$ of $v^\perp$, and let g be the matrix with rows $x_1, ..., x_{n-1}, v$. These vectors are mutually orthogonal with unit norm, so $g \in O(n)$. If $\det g = -1$, then scale x_1 by -1. Then $g \in \mathrm{SO}(n)$, and $vg = e_n$ as claimed.

The isotropy group of the standard basis vector e_n is easily computed

$$\mathrm{SO}(n)_{e_n} = \{g \in \mathrm{SO}(n) : g = \begin{bmatrix} h & 0 \\ 0 & 1 \end{bmatrix} \text{ as a block matrix, where } h \in \mathrm{SO}(n-1)\}.$$

Thus, the sphere S is a model of the quotient

$$S \approx \mathrm{SO}(n-1) \setminus \mathrm{SO}(n).$$

Where $\mathrm{SO}(n-1) \setminus \mathrm{SO}(n)$ is a *right*-SO(n) space, under the action $\mathrm{SO}(n-1)\gamma \times g \longmapsto \mathrm{SO}(n-1)\gamma g$.

This isomorphism can be made tangible by noticing each coset $\mathrm{SO}(n-1)g$ is uniquely distinguished by its bottom row

$$\mathrm{SO}(n-1)g = \mathrm{SO}(n-1) \begin{bmatrix} * & * & \cdots & * & * \\ * & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & * \\ v_1 & v_2 & \cdots & v_{n-1} & v_n \end{bmatrix}$$

with $(v_1, .., v_n)$ a vector of unit norm.

The Lie algebra $\mathfrak{so}_n$ parametrizes the vectors tangent to the identity in SO(n) via the exponential map,

$$\mathfrak{so}_n = \text{ real } n \times n \text{ matrices } X \text{ such that } e^{tX} \in \mathrm{SO}(n) \text{ for all } t \in \mathbb{R}.$$

The Lie algebra acts on smooth functions $f : \mathrm{SO}(n) \to \mathbb{C}$ via

$$X \cdot f(g) = \frac{d}{dt}\bigg|_0 f(ge^{tX}).$$

Granting this, we can compute an explicit description of $\mathfrak{so}_n$ via the characterization

$$\mathrm{SO}(n) = n \times n \text{ matrices } g \text{ of determinant 1 such that } gg^\top = 1.$$

Indeed, compute for $X \in \mathfrak{so}_n$

$$(e^{-tX}) = (e^{tX})^{-1} = (e^{tX})^\top = e^{tX^\top},$$

and differentiate the expression at $t = 0$,

$$-X = \frac{d}{dt}\bigg|_0 e^{-tX} = \frac{d}{dt}\bigg| e^{tX^\top} = X^\top$$

demonstrating that X is skew symmetric. Conversely, if X is skew symmetric then

$$(e^{tX})^\top = e^{tX^\top} = e^{-tX} = (e^{tX})^{-1}$$

showing that $X \in \mathfrak{so}_n$. Thus

$$\mathfrak{so}_n = \text{ real vectorspace of } n \times n \text{ skew symmetric matrices.}$$

Now let e_{ij} be the $n \times n$ matrix with a 1 in the ith row and jth column, and 0 everywhere else. Then $\mathfrak{so}_n$ is spanned by the collection

$$\{X_{ij}\} = \{e_{ij} - e_{ji} : 1 \leq i, j \leq n\}.$$

The subset $\{X_{ij} : 1 \leq i < j \leq n\}$ is a *basis*.

Remark 2. Despite our general agenda of working intrinsically rather than in coordinates, the reader who is unfamiliar with standard Lie theoretic computations could profit from performing these computations explicitly, for low dimensions.

For example, this basis of $\mathfrak{so}(3)$ is

$$X_{12} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \quad X_{23} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

which exponentiate to

$$e^{tX_{12}} = \begin{bmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad e^{tX_{13}} = \begin{bmatrix} \cos t & 0 & \sin t \\ 0 & 1 & 0 \\ -\sin t & 0 & \cos t \end{bmatrix}, \quad e^{tX_{23}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{bmatrix}.$$

Now consider a suitably differentiable function[1] f on S^2 and compute

$$\begin{aligned} X_{12} f(x_1, x_2, x_3) &= \frac{d}{dt}|_{t=0} f(x_1 \cos t + x_2 \sin t, -x_1 \sin t + x_2 \cos t, x_3) \\ &= (-x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}) f(x_1, x_2, x_3) \end{aligned}$$

and similar computations show

$$X_{13} = -x_3 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_3}, \quad X_{23} = -x_3 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3}$$

as operators. More generally $X_{ij} \in \mathfrak{so}_n$, acts on smooth functions on S by $-x_i \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial x_i}$. One should note that of the $\binom{n}{2}$ operators X_{ij}, only $n-1$ are linearly independent at any point on the sphere.

[1] Some care must be taken in this computation. For a function defined only on f, symbols such as $\frac{\partial}{\partial x}$ are not immediately sensible everywhere on the sphere. However, for a function f on S, define a function F on $\mathbb{R}^n$ by $F(x) = f(x/|x|)$. Then one may interpret partial derivatives of f as partial derivatives of F, restricted to S.

3.2.2 Invariant operators on S

Since S is a model of the quotient $\mathrm{SO}(n-1)\backslash\mathrm{SO}(n)$ we may identify functions on the former with *left*-$\mathrm{SO}(n-1)$ invariant functions on $\mathrm{SO}(n)$. Observe that left-$\mathrm{SO}(n-1)$ invariance of a function $f : \mathrm{SO}(n) \to \mathbb{C}$ is preserved by the action of $\mathfrak{so}_n$,

$$\begin{aligned} ((X \cdot f)(kg) &= \frac{d}{dt}\Big|_0 f(kge^{tx}) \\ &= \frac{d}{dt}\Big|_0 f(ge^{tx}) = (X \cdot f)(g) \quad \text{for } X \in \mathfrak{so}_n. \end{aligned}$$

Thus, we define the complex vectorspace

$$C^\infty(S) = \text{smooth left-SO}(n-1) \text{ invariant functions on SO}(n).$$

Invariant integral

As a compact[2] topological group, $\mathrm{SO}(n)$ is equipped with a invariant measure $\mathrm{d}\mu$. Since $\mathrm{SO}(n)$ and $\mathrm{SO}(n-1)$ are compact, they are both unimodular, so the invariant measure on $\mathrm{SO}(n)$ descends to an invariant measure on the quotient $\mathrm{SO}(n-1) \setminus \mathrm{SO}(n)$, still denoted $\mathrm{d}\mu$. The integral is characterized by (letting $\overline{g} = \mathrm{SO}(n-1)g$)

$$\int_{\mathrm{SO}(n)} f(g)\,\mathrm{d}g = \int_{\mathrm{SO}(n)\backslash\mathrm{SO}(n-1)} \int_{\mathrm{SO}(n-1)} f(kg)\,\mathrm{d}k\,\mathrm{d}\mu(\overline{g}) \quad \text{for all } f \in C^\infty(\mathrm{SO}(n)).$$

Note that $\mathrm{d}\mu$ is still invariant with respect to the right action of $\mathrm{SO}(n)$ on $\mathrm{SO}(n-1) \setminus \mathrm{SO}(n)$. Define a Hermitian inner product on $C^\infty(S)$ by

$$\langle f, g\rangle = \int_{\mathrm{SO}(n-1)\backslash\mathrm{SO}(n)} f\overline{g}\,\mathrm{d}\mu = \int_S f\overline{g}\,\mathrm{d}\mu.$$

Then define

$$L^2(S) = \text{ completion of } C^\infty(S) \text{ with respect to the norm } |f|^2 = \int_S |f|^2\,\mathrm{d}\mu$$

Invariant derivative

In general, differentiation on S does not commute with the right action of $\mathrm{SO}(n)$. Indeed, compute for $x \in \mathfrak{so}_n$ with corresponding vector field X, and any $h \in \mathrm{SO}(n)$

$$X \cdot f(gh) = \frac{d}{dt}|_{t=0} f(ghe^{tx}),$$

[2]Note that existence of a left Haar measure and a right Haar measure requires only that the group be a *locally* compact topological group. On compact groups, the Haar measure is left-and-right invariant.

showing that commutativity would imply $h^{-1}xh$ for all $h \in \mathrm{SO}(n)$, which is not true for any $x \in \mathfrak{so}_n$.

The simplest nonzero invariant differential operator is the *Casimir* element, denoted Ω, coming from the universal enveloping algebra of $\mathfrak{so}_n$. For details, see [18].

Equip $\mathfrak{so}_n$ with the non-degenerate bilinear pairing

$$\langle A, B\rangle = \mathrm{Trace}(AB^\top),$$

according to which, the dual element X_{ij}^* is $\frac{1}{2}X_{ij}$. Thus, up to a constant, the Casimir element for $\mathfrak{so}_n$ is

$$\Omega = \sum_{1\leq i<j\leq n} X_{ij}^2.$$

The Casimir operator commutes with the left and right action of $\mathrm{SO}(n)$, by design.

Define the **spherical Laplacian**

$$\Delta = \Delta^S = \text{ restriction of } \Omega \text{ to } C^\infty(S).$$

Remark 3. For reference, in cartesian coordinates, the spherical Laplacian is

$$\begin{aligned}\Delta^S &= \sum_{i<j}(x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i})^2\\ &= \sum_{i<j} x_i^2\frac{\partial^2}{\partial x_j^2} - x_i\frac{\partial}{\partial x_i} - 2x_ix_j\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_j} + x_j^2\frac{\partial^2}{\partial x_i^2}.\end{aligned}$$

Let $E = \sum_{i=1}^n x_i\frac{\partial}{\partial x_i}$ be the Eulerian operator. Leibniz's identity is $\frac{\partial}{\partial x_i}x_i = 1 + x_i\frac{\partial}{\partial x_i}$, from which we see $(x_i\frac{\partial}{\partial x_i})^2 = x_i\frac{\partial}{\partial x_i}x_i\frac{\partial}{\partial x_i} = x_i\frac{\partial}{\partial x_i} + x_i^2\frac{\partial^2}{\partial x_i^2}$, and thus

$$E^2 = (\sum_{i=1}^n x_i\frac{\partial}{\partial x_i})^2 = \sum_{i=1}^n x_i^2\frac{\partial^2}{\partial x_i^2} + 2\sum_{i<j} x_ix_j\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} + E.$$

Now compute, recognizing $E^2 - \sum_{i=1}^n x_i^2\frac{\partial^2}{\partial x_i^2} - E$ in the third equality,

$$\begin{aligned}\Delta^S &= \sum_{i<j}(x_i\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_i})^2\\ &= \sum_{i<j}\left(x_i^2\frac{\partial^2}{\partial x_j^2} - x_i\frac{\partial}{\partial x_i} - 2x_ix_j\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} - x_j\frac{\partial}{\partial x_j} + x_j^2\frac{\partial^2}{\partial x_i^2}\right)\\ &= \sum_{i=1}^n(r^2 - x_i^2)\frac{\partial^2}{\partial x_j^2} - (n-1)E - E^2 + \sum_{i=1}^n x_i^2\frac{\partial^2}{\partial x_i^2} - E\\ &= r^2\Delta^{\mathbb{R}^n} - (n-2)E - E^2.\end{aligned}$$

Thus

$$\Delta^S = \Delta^{\mathbb{R}^n} - E(E+n-2).$$

Verifications: positivity and symmetry

In order apply spectral theory to $L^2(S)$, we need to verify that Δ is symmetric and negative on $C^\infty(S)$, with respect to the invariant inner product $\langle \cdot, \cdot \rangle$. Compute for $f, F \in C^\infty(S)$ and $X_{ij} \in \mathfrak{so}_n$, using invariance of $\mathrm{d}\mu$,

$$\int_S f(ge^{tX_{ij}})\overline{F}(g)\,\mathrm{d}\mu(g) = \int_S f(g)\overline{F}(ge^{-tX_{ij}})\,\mathrm{d}\mu(g).$$

Then taking derivatives of both sides, citing that f and F are smooth functions on a compact space to pass the derivative through the integral,

$$\begin{aligned}\int_S \frac{d}{dt}|_{t=0} f(ge^{tX_{ij}})\overline{F}(g)\,\mathrm{d}\mu(g) &= \int_S f(g)\frac{d}{dt}|_{t=0}\overline{F}(ge^{-tX_{ij}})\,\mathrm{d}\mu(g)\\ &= -\int_S f(g)\frac{d}{dt}|_{t=0}\overline{F}(ge^{tX_{ij}})\,\mathrm{d}\mu(g),\end{aligned}$$

so

$$\langle X_{ij} \cdot f, F\rangle = -\langle f, X_{ij} \cdot F\rangle.$$

Repeating the computation shows that

$$\langle X_{ij}^2 \cdot f, F\rangle = \langle f, X_{ij}^2 \cdot F\rangle \quad \text{for all } f, F \in C^\infty(S)$$

proving each X_{ij}^2 is a symmetric operator on $C^\infty(S)$. Moreover, for each $f \in C^\infty(S)$,

$$\langle X_{ij}^2 \cdot f, f\rangle = -\langle X_{ij} \cdot f, X_{ij} \cdot f\rangle = -|X_{ij} f|^2 \le 0$$

showing that each X_{ij}^2 is *negative*.

Thus their sum, Δ^S, is a symmetric, negative, (unbounded) operator on $L^2(S)$, with dense domain $C^\infty(S)$.

3.2.3 Functions on S

By Friedrichs' construction, the symmetric, positive, densely defined operator $-\Delta$ on $C^\infty(S)$ has a positive, *self-adjoint* extension $-\widetilde{\Delta}$ defined on a dense subspace the $+1$ Sobolev space

$$H^1(S) = \text{completion of } C^\infty(S) \text{ with respect to } |f|_{H^1}^2 = \langle (1-\Delta)f, f\rangle.$$

The resolvent $(1-\widetilde{\Delta})^{-1} : L^2(S) \to H^1(S)$ is continuous, positive, self-adjoint, and satisfies

$$\langle f, F\rangle = \langle (1-\Delta)f, (1-\widetilde{\Delta})^{-1}F\rangle \quad \text{for } f \in C^\infty(S) \text{ and } F \in L^2(S).$$

Compactness of the inclusion $H^1(S) \longrightarrow L^2(S)$

To show that $(1-\widetilde{\Delta})^{-1}$ is compact, it suffices to prove that the inclusion $H^1(S) \to L^2(S)$ is compact. Granting compactness of the resolvent, the spectral theorem provides an orthonormal basis of $L^2(S)$ consisting of eigenfunctions of $(1-\widetilde{\Delta})^{-1}$.

Claim 5. The inclusion $H^1(S) \to L^2(S)$ is compact.

Proof. This proof serves as an archetype for compactness arguments on compact Riemannian manifolds, as in [15]. The technique is to reduce a function on the sphere to a collection of functions on a region in $\mathbb{R}^{n-1}$ by means of a smooth partition of unity subordinate to a finite collection of coordinate patches. Since the sphere is compact, the coordinate patches have compact closure in $\mathbb{R}^{n-1}$ so the corresponding truncated functions can be identified with functions on $\mathbb{T}^{n-1} = \mathbb{R}^{n-1}/\mathbb{Z}^{n-1}$, where we know Rellich compactness holds. Some care must be taken to make sure that the $H^1(S)$ and $L^2(S)$ norms are respected by their counterparts on flat tori.

Fix $\varepsilon \in (1/\sqrt{n}, 1)$, and define a neighborhood of the identity in $\mathrm{SO}(n)$

$$U = \{g \in \mathrm{SO}(n) : g_{nn} > 1 - \varepsilon\},$$

and let $V = \mathrm{SO}(n-1)U$. Identify V with a neighborhood of $e_n = (0, ..., 0, 1)$ in S. If $g_i^{\pm}$ is any rotation that takes e_n to $\pm e_i$, then the $2n$ translates Vg_i cover S. Relabel so that i runs through $1, ..., 2n$, and let $\{\varphi_1, .., \varphi_{2n}\}$ be a smooth partition of unity subordinate to this subcover.

For each i and smooth function $f : S \to \mathbb{C}$, certainly

$$|\varphi_i f|_{L^2(S)} \leq |f|_{L^2(S)},$$

so

$$\frac{1}{2n} \sum_{i=1}^{2n} |\varphi_i f|_{L^2(S)} \leq |f|_{L^2(S)}.$$

On the other hand, by Cauchy–Schwarz–Bunyakowski,

$$|f|^2_{L^2(S)} = \langle f, \sum_{i=1}^{2n} \varphi_i f\rangle = \sum_{i=1}^{2n} \langle f, \varphi_i f\rangle \leq |f|_{L^2(S)} \sum_{i=1}^{2n} |\varphi_i f|_{L^2(S)}.$$

so $|f|_{L^2(S)} \leq \sum_{i=1}^{2n} |\varphi_i f|_{L^2(S)}$. In sum, we have

$$\frac{1}{2n} \sum_{i=1}^{2n} |\varphi_i f|_{L^2(S)} \leq |f|_{L^2(S)} \leq \sum_{i=1}^{2n} |\varphi_i f|_{L^2(S)}.$$

Letting $C^\infty(Vg_i)$ subspace of $C^\infty(S)$ consisting of functions supported on Vg_i, define $L^2(Vg_i)$ to be the completion of $C^\infty(Vg_i)$ with respect to $|\cdot|_{L^2}$. We have shown that the map

$$L^2(S^{n-1}) \longrightarrow \bigoplus L^2(Vg_i) \qquad f \longmapsto (\varphi_1 f, ..., \varphi_{2n} f)$$

is an isomorphism to its image.

To prove a similar reduction for $H^1(S)$, we must ensure that differentiation of the partition of unity does not interfere with the norm comparisons. For smooth $f : S \longrightarrow \mathbb{C}$, compute using the skew-adjointness of $X_{ij} \in \mathfrak{so}_n$, and that $|\varphi_i| \leq 1$,

$$\begin{aligned} |\varphi_i f|^2_{H^1(S)} &= \langle (1-\Delta)\varphi_i f, \varphi_i f\rangle \\ &= |\varphi_i f|^2_{L^2(S)} + \sum_{1\leq j<k\leq n} |X_{jk}\varphi_i f|^2_{L^2} \\ &\leq |f|^2_{L^2(S)} + \sum_{1\leq j<k\leq n} |X_{jk}\varphi_i f|^2_{L^2} \end{aligned}$$

Since φ_i is smooth, each $X_{jk}\varphi_i$ is smooth for $1 \leq j < k \leq n$. Since the sphere is compact, each $X_{jk}\varphi_i$ is bounded. Thus, there is some constant A independent of f that uniformly bounds all $X_{ij}\varphi_i$. With no loss of generality, assume $A \geq 1$.

Writing X for a generic X_{jk}, and φ for φ_i, compute

$$\begin{aligned} |X(\varphi f)|_{L^2} = |X\varphi \cdot f + \varphi X f|_{L^2} &\leq |fX\varphi|_{L^2} + |\varphi X f|_{L^2} \\ &\leq A(|f|_{L^2} + |Xf|_{L^2}) \leq A|f|_{H^1(S)}. \end{aligned}$$

Thus, after summing and replacing $\binom{n}{2}A$ by A we see

$$|\varphi_i f|^2_{H^1(S)} \leq |f|_{L^2(S)} + A|f|^2_{H^1(S)} \leq 2A|f|^2_{H^1(S)}.$$

After summing over i, we see

$$\frac{1}{4nA}\sum_{i=1}^{2n} |\varphi_i f|_{H^1(S)} \leq |f|_{H^1(S)}$$

For the other comparison, compute using Cauchy–Schwarz–Bunyakowski

$$|f|^2_{H^1(S)} = \langle f, \sum_{i=1}^{2n} \varphi_i f\rangle_{H^1(S)} \leq |f|_{H^1(S)} \sum_{i=1}^{2n} |\varphi_i f|_{H^1(S)}.$$

Clearing the common factor of $|f|_{H^1(S)}$ gives $|f|_{H^1(S)} \leq \sum_{i=1}^{2n} |\varphi_i f|_{H^1(S)}$.

Now combine the comparisons

$$\frac{1}{4nA}\sum_{i=1}^{2n} |\varphi_i f|_{H^1(S)} \leq |f|_{H^1(S)} \leq \sum_{i=1}^{2n} |\varphi_i f|_{H^1(S)}$$

to demonstrate that the map

$$H^1(S) \approx \bigoplus_{i=1}^{2n} H^1(Vg_i) \quad f \longmapsto (\varphi_1 f, ..., \varphi_{2n} f),$$

is an isomorphism to its image.

Recall that $V = \{\mathrm{SO}(n-1)g \in \mathrm{SO}(n-1)\setminus\mathrm{SO}(n) : g_{nn} > 1-\varepsilon\}$ with $\varepsilon \in (1-1/\sqrt{n}, 1)$. Define the coordinate map

$$\psi : V \longrightarrow D \subset \mathbb{R}^{n-1} \quad (x_1, ..., x_n) \longmapsto (x_1, ..., x_{n-1}) = \xi.$$

by projection onto the open equatorial ε-disk D in $\mathbb{R}^{n-1}$. Since $\varepsilon < 1$, the map ψ extends continuously to the closure of D, which does not include the equator $x_n = 0$. The inverse (parametrization) map is

$$\psi^{-1} = \Psi : D \longrightarrow V \subset S \quad \xi \longmapsto (\xi, (1 - |\xi|^2)^{1/2}).$$

Abbreviate $(1 - |\xi|^2)^{1/2}$ to ξ_n.

Identify D with an open set in $\mathbb{T}^{n-1}$, let $C^\infty(D)$ be the subspace of functions in $C^\infty(\mathbb{T}^{n-1})$ supported on D. Let $L^2(D)$ be the completion of $C^\infty(D)$ with respect to $|\cdot|_{L^2(\mathbb{T}^{n-1})}$, and likewise define $H^1(D)$ as the completion of $C^\infty(D)$ with respect to $|\cdot|_{H^1(\mathbb{T}^{n-1})}$. Identify $L^2(D)$ and $H^1(D)$ with subspaces of $L^2(\mathbb{T}^{n-1})$ and $H^1(\mathbb{T}^{n-1})$ respectively. By the first section, we know $H^1(\mathbb{T}^{n-1}) \to L^2(\mathbb{T}^{n-1})$ is compact, and by restriction $H^1(D) \to L^2(D)$ is compact.

To prove that the inclusion $H^1(S) \longrightarrow L^2(S)$ is compact, it suffices to prove the local claim that $H^1(V) \longrightarrow L^2(V)$ is compact. To do this, we show $H^1(V)$ is isomorphic to its image in $H^1(D)$ and that $L^2(V)$ is isomorphic to its image in $L^2(D)$. Rellich's lemma shows that the bottom map in the following diagram is compact

$$\begin{array}{ccc} H^1(V) & \dashrightarrow & L^2(V) \\ \downarrow & & \uparrow \\ H^1(D) & \xrightarrow[\text{compact}]{} & L^2(D) \end{array}$$

where the dotted arrow along the top connotes traversal along the other three sides. Since the dotted arrow is the composition of an isomorphism (to its image), a compact inclusion, and an isomorphism (on a subspace), it is compact, proving the claim locally.

To facilitate change of variables in the relevant integrals, compute

$$\Psi' = \begin{pmatrix} \mathrm{id}_{n-1} \\ \\ \xi_n^{-1}\xi \end{pmatrix} \quad \text{so} \quad \sqrt{\det \Psi'^\top \Psi'} = \xi_n^{-1}$$

Observe that on the closure of D, where $1 \leq \xi_n \leq 1 - \varepsilon$, the spherical volume dilation factor satisfies $1 \leq \sqrt{\det \Psi' \Psi'^\top} \leq \frac{1}{1-\varepsilon}$.

A smooth function $f : S \to \mathbb{C}$ gives rise to a function $f \circ \Psi = F : D \to \mathbb{C}$. Compute using the upper bound on the dilation factor, (recalling that $\xi_n = \sqrt{\xi_1^2 + ... + \xi_{n-1}^2}$)

$$\begin{aligned} |f|_{L^2(V)}^2 &= \int_V |f(g)|^2 \, \mathrm{d}\mu(g) \\ &= \int_{\Psi(D)} |(F \circ \psi)(g)|^2 \, \mathrm{d}\mu(g) \\ &= \int_D |F(\xi)|^2 \cdot \xi_n^{-1} \, \mathrm{d}\xi \\ &\leq \frac{1}{1-\varepsilon} \int_D |F(\xi)|^2 \, \mathrm{d}\xi = \frac{1}{1-\varepsilon} |F|_{L^2(\mathbb{T}^{n-1})}^2, \end{aligned}$$

and using the lower bound,

$$\begin{aligned} |F|^2_{L^2(\mathbb{T}^{n-1})} &= \int_D |F(\xi)|^2 \,\mathrm{d}\xi \\ &\leq \int_D |F(\xi)|^2 \cdot \xi_n^{-1} \,\mathrm{d}\xi = |f|_{L^2(V)}. \end{aligned}$$

Thus

$$(1-\varepsilon)|f|_{L^2(V)} \leq |F|_{L^2(\mathbb{T}^{n-1})} \leq |f|_{L^2(V)},$$

giving the first isomorphism

$$L^2(V) \approx L^2(D) \quad f \longmapsto F.$$

To establish the isomorphism of Sobolev spaces we show that the maps

$$H^1(V) \longrightarrow H^1(D) \quad f \longmapsto f \circ \Psi$$

and its inverse

$$H^1(D) \longrightarrow H^1(V) \quad F \longmapsto F \circ \psi$$

are continuous.

We have the result for the L^2 part of the norms. For the other part, first compute

$$\begin{aligned} \langle -\Delta^{\mathbb{T}^{n-1}}(f\circ\Psi), f\circ\Psi\rangle_{L^2(D)} &= \sum_{i=1}^{n} |(f\circ\Psi)_i|^2_{L^2(D)} \\ &= \int_D \sum_{i=1}^{n-1} \left(f_i(\xi,\xi_n) - \xi_i\xi_n^{-1} f_n(\xi,\xi_n)\right)^2 \,\mathrm{d}\xi. \end{aligned}$$

Second, compute on the sphere

$$\begin{aligned} \langle -\Delta^{S^{n-1}} f, f\rangle_{L^2(V)} &= \sum_{i<j} \langle X_{ij} f, X_{ij} f\rangle \\ &= \sum_{i<j} |x_i f_j - x_j f_i|^2_{L^2(V)} \\ &= \sum_{1\leq i<j\leq n-1} |x_i f_j - x_j f_i|^2_{L^2(V)} + \sum_{i=1}^{n-1} |x_n f_i - x_i f_n|_{L^2(V)} \\ &\geq \sum_{i=1}^{n-1} |x_n f_i - x_i f_n|_{L^2(V)} \end{aligned}$$

Pull back to integrate the latter on the torus,

$$\begin{aligned} \langle -\Delta^{S^{n-1}} f, f\rangle &\geq \int_D \sum_{i=1}^{n-1} (\xi_n f_i(\xi,\xi_n) - \xi_i f_n(\xi,\xi_n))^2\, \xi_n^{-1} \,\mathrm{d}\xi \\ &= \int_D \sum_{i=1}^{n-1} \xi_n \left(f_i(\xi,\xi_n) - \xi_i\xi_n^{-1} f_n(\xi,\xi_n)\right)^2 \,\mathrm{d}\xi \\ &\geq \frac{1}{1-\sqrt{n}} \langle -\Delta^{\mathbb{T}^{n-1}}(f\circ\Psi), f\circ\Psi\rangle_{L^2(D)} \end{aligned}$$

since ξ_n is at least $1-\sqrt{n}$. Thus,

$$|f\circ\Psi|_{H^1(D)} \le |f|_{H^1(V)}$$

making the map $f \longmapsto f\circ\Psi$ from $H^1(V)$ to $H^1(D)$ continuous.

To show that the inverse is continuous, let F be smooth on D, so that $F\circ\psi$ on V is $F\circ\psi(x_1,...,x_n) = F(x_1,...,x_{n-1})$. Again, the L^2 part of the Sobolev norms have already been compared. For the other part, compute noting that $X_{in}(F\circ\psi)=0$, and that $|x_i|<\varepsilon$

$$\begin{aligned}
\sum_{i<j} |X_{ij}(F\circ\psi)|^2_{L^2(V)} &= \sum_{1\le i<j\le n-1} \int_V (x_j F_i(x_1,..,x_{n-1}) - x_i F_j(x_1,..,x_{n-1}))^2 \,\mathrm{d}\mu(x) \\
&\le \sum_{1\le i<j\le n-1} \int_V (\varepsilon|F_i(x_1,..,x_{n-1})| + \varepsilon|F_j(x_1,..,x_{n-1})|)^2 \,\mathrm{d}\mu(x) \\
&\le 4n^2\varepsilon^2 \sum_{i=1}^{n} \int_V (|F_i(x_1,..,x_{n-1})|)^2 \,\mathrm{d}\mu(x) \\
&= 4n^2\varepsilon^2 \sum_{i=1}^{n} \int_D |F_i(\xi)|^2 \xi_n^{-1} \,\mathrm{d}\xi \\
&\le 4n^2 \frac{\varepsilon^2}{1-\varepsilon} \sum_{i=1}^{n} \int_D F_i(\xi)^2 \,\mathrm{d}\xi.
\end{aligned}$$

Thus, taking into account the already computed L^2 parts of the norms, we have

$$\frac{\sqrt{1-\varepsilon}}{2n\varepsilon} |F\circ\psi|_{H^1(V)} \le |F|_{H^1(D)},$$

showing that $H^1(D) \longrightarrow H^1(V)$ is continuous.

Thus, the map $H^1(V)\to H^1(D)$ given by $f\longmapsto f\circ\Psi$ is a Hilbert space isomorphism,

$$H^1(V) \approx H^1(D).$$

The inclusion $H^1(V)\longrightarrow L^2(V)$ is equivalently the composition

$$H^1(V) \xrightarrow{\sim} H^1(D) \xrightarrow{\text{compact}} L^2(D) \xrightarrow{\sim} L^2(V),$$

which is compact. Since $H^1(S)$ and $L^2(S)$ are identified with a subspace of finite direct product of $H^1(V)$ and $L^2(V)$ respectively, the inclusion $H^1(S)\longrightarrow L^2(S)$ is compact. □

Consequently, $L^2(S)$ has an orthonormal basis of eigenfunctions for $(1-\widetilde{\Delta^S})^{-1}$, which are eigenfunctions of $\widetilde{\Delta^S}$.

Sobolev regularity

To prove that the eigenfunctions of $\widetilde{\Delta^S}$ coincide with those of Δ^S, it suffices to prove that they are smooth, so that evaluation of the extension coincides with evaluation of the original.

Let $f \in H^1(S)$ be an eigenfunction of $\widetilde{\Delta}^S$ with eigenvalue $\lambda \in \mathbb{R}_{\leq 0}$. Then

$$(1-\widetilde{\Delta}^S)f = (1-\lambda)f.$$

Consequently,

$$f = (1-\widetilde{\Delta}^S)^{-1}(1-\lambda)f \in (1-\widetilde{\Delta}^S)^{-1}H^1(S).$$

So, by induction

$$f \in (1-\widetilde{\Delta}^S)^{-k}H^1(S) \quad \text{for all } k \in \mathbb{Z}_{\geq 0}.$$

Defining the $+k$ Sobolev space

$$H^k(S) = \text{completion of } C^\infty(S) \text{ with respect to } |f|^2_{H^k(S)} = \langle (1-\Delta^S)^k f, f\rangle,$$

the penultimate display shows that the eigenfunction f is in $H^k(S)$ for all $k \in \mathbb{Z}_{\geq 0}$. That is, f lives in

$$H^\infty(S) = \bigcap_{k\in\mathbb{Z}_{\geq 0}} H^k(S) = \lim_{\leftarrow} H^k(S^{n-1}).$$

To prove that F is smooth, it suffices to show that $H^\infty(S) = C^\infty(S)$.

Claim 6. $H^\infty(S) = C^\infty(S)$

Proof. To simplify notation, define a 'vector' of first order differential operators on S

$$X = (X_{ij}) \quad \text{ for } 1 \leq i < j \leq n$$

For a multi-index $(\alpha_{ij}) \in \mathbb{Z}_{\geq 0}^{n(n-1)/2}$, let $X^\alpha = \prod_{i<j} X_{ij}^{\alpha_{ij}}$, and let $|\alpha| = \sum_{1\leq i<j\leq n} \alpha_{ij}$.

Define a family of Banach spaces

$$C^k(S) = \text{ completion of } C^\infty(S) \text{ with respect to } |f|_{C^k} = \sum_{\ell=0}^{n} \sup_{|\alpha|\leq \ell} \sup_{x\in S} |X^\alpha f(x)|.$$

By definition,

$$C^\infty(S) = \bigcap_{k\in\mathbb{Z}_{\geq 0}} C^k(S) = \lim_{\leftarrow} C^k(S).$$

To prove that $C^\infty(S) = H^\infty(S)$, it suffices to show that for some constants a, b

$$a|f|_{H^k} \leq |f|_{C^k} \leq b|f|_{H^{k+1}} \quad \text{ for all smooth } f.$$

Expanding the H^k norm, using the binomial theorem, dominating by the central coefficient, we see

$$|f|^2_{H^k} = \langle (1-\Delta^S)^k f, f\rangle \leq \frac{2n!}{(n!)^2} \sum_{\ell=0}^{k} \sum_{|\alpha|\leq \ell} |X^\alpha f|^2_{L^2}.$$

Since the sphere is compact, certainly

$$|X_{ij}^{\ell} f|_{L^2}^2 \leq \mathrm{vol}(S) \sup_{x\in S} |X_{ij}^{\ell} f(x)|^2 \leq \mathrm{vol}(S) \sup_{|\alpha|\leq \ell} \sup_{x\in S} |X^{\alpha} f(x)|^2,$$

so

$$|f|_{H^k}^2 \leq \frac{\mathrm{vol}(S)2n!}{(n!)^2} \sum_{\ell=0}^{k} \binom{n}{2} \sup_{|\alpha|\leq \ell} \sup_{x\in S} |X^{\alpha} f(x)|^2 = \frac{\mathrm{vol}(S)2n!}{(n!)^2} \binom{n}{2} |f|_{C^k}^2.$$

Thus, the continuous inclusions in the limitands extends to a continuous inclusion of limits $C^{\infty}(S) \to H^{\infty}(S)$.

To prove the other containment, we use the Sobolev inequality on $\mathbb{T}^{n-1}$: for any integer ℓ such that $\ell > k + (n-1)/2$ and $f \in C^{\infty}(\mathbb{T}^{n-1})$, we have

$$|f|_{C^k(\mathbb{T}^{n-1})} \leq C|f|_{H^{\ell}(\mathbb{T}^{n-1})} \quad \text{with } C \text{ independent of } f.$$

For a proof, see [30].

Fix k and let $\ell > k + (n-1)/2$. We prove the local claim that the H^{ℓ} norm dominates the C^k norm for functions supported on the neighborhood V of e_n (as defined in the last section). Since $H^{\ell}(S)$ and $C^k(S)$ isomorphic to subspace of the direct sum of finitely many of their local counterparts $H^{\ell}(V)$ and $C^k(V)$ respectively, this proves the global domination.

From the proof in the previous section, we know that

$$H^1(V) \approx H^1(D)$$

where V is a neighborhood of e_n and D is an open subset of the equatorial disk $e_n = 0$. Induction shows that $H^{\ell}(V) \approx H^{\ell}(D)$.

Identified with a subspace of $H^{\ell}(\mathbb{T}^{n-1})$, the Sobolev inequality shows that $H^{\ell}(D)$ imbeds continuously in $C^k(D)$, identified with a subspace of $C^k(\mathbb{T}^n)$. Further, by induction, we have

$$C^k(D) \approx C^k(V) \quad \text{for every } k.$$

Consequently, by the Sobolev embedding on $\mathbb{T}^{n-1}$, we see that $H^{\ell}(S) \longrightarrow C^k(S)$ is continuous whenever $\ell > k + (n-1)/2$. This gives rise to a continuous embedding of limits $H^{\infty}(S) \to C^{\infty}(S)$, proving the claim. □

Consequently, every eigenfunction $\widetilde{\Delta}^S$ is an eigenfunction of Δ^S, showing that $L^2(S)$ has an orthonormal basis of eigenfunctions for Δ^S.

3.2.4 Complete determination of eigenfunctions of Δ^S

In this section, we show that the eigenfunctions of Δ^S are the *harmonic, homogeneous polynomials* on $\mathbb{R}^n$, restricted to S.

Harmonic homogeneous functions restrict to eigenfunctions

Recall the formula for the spherical Laplacian

$$\Delta^S = \sum_{i<j} X_{ij}^2 = \Delta^{\mathbb{R}^n} - E(E+n-2)$$

where $E = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$ is the Eulerian operator.

First, we prove a useful identity for E. Recall that a positive homogeneous function $f : \mathbb{R}^n - \{0\} \to \mathbb{C}$ of degree s satisfies

$$f(tx) = t^s f(x) \quad \text{for } t \in \mathbb{R}_{\geq 0}.$$

For such an f define the function $g : \mathbb{R}_{\geq 0} \to \mathbb{C}$ by $g(t) = f(tx)$ for a fixed $x \in \mathbb{R}^n - \{0\}$. Compute on one hand, using the chain rule

$$\frac{d}{dt} g(t) = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} f(tx) = Ef(tx),$$

and on the other, using positive homogeneity

$$\frac{d}{dt} g(t) = \frac{d}{dt} t^s f(x) = st^{s-1} f(x).$$

Evaluating at $t = 1$ shows

$$Ef = sf \quad \text{(for } f \text{ positive homogeneous of degree } s\text{)}.$$

Thus, restricting such an f to the sphere, we see

$$\Delta^S f = \Delta^{\mathbb{R}^n} f + (-s^2 + (2-n)s) f.$$

In particular, if f is also *harmonic*, meaning $\Delta^{\mathbb{R}^n} f = 0$, then then the restriction of f to S is an eigenfunction of Δ^S, with eigenvalue $-s(s+n-2)$.

Eigenfunctions are restrictions of harmonic homogeneous functions

Conversely, any eigenfunction of Δ^S arises as the restriction of a *harmonic* positive homogeneous function, as follows. Suppose $f : S \to \mathbb{C}$ is an eigenfunction of Δ^S, with eigenvalue λ. Define $g : \mathbb{R}^n - \{0\} \to \mathbb{C}$, the degree zero positive homogeneous extension of f,

$$g(x) = f(x/|x|).$$

Note that $g|_S = f$, and that $(\Delta^{\mathbb{R}^n} g)|_S = \Delta^S f = \lambda f$.

For any (potentially complex) value s satisfying $\lambda = -s(s+n-2)$ define the degree s positive homogeneous function $F : \mathbb{R}^n - \{0\} \to \mathbb{C}$ by

$$F(x) = |x|^s g(x).$$

The assertion is that F is harmonic.

To this end, compute $\Delta^{\mathbb{R}^n}(|x|^{-s}F) = \Delta^{\mathbb{R}^n} g$. First,

$$\begin{aligned}\frac{\partial^2}{\partial x_i^2}(|x|^{-s}F) =& \frac{\partial}{\partial x_i}(-sx_i|x|^{-s-2}F + |x|^{-s}F_i)\\ =& -s(|x|^{-(s+2)} - (s+2)x_i^2|x|^{-(s+4)})F - s|x|^{-(s+2)}x_iF_i\\ &- s|x|^{-(s+2)}x_iF_i + |x|^{-s}F_{ii}\\ =& -s|x|^{-(s+2)}((1-(s+2)x_i^2|x|^{-2})F + 2x_iF_i) + |x|^{-s}F_{ii}.\end{aligned}$$

Then summing over i, we see (where E is the Eulerian operator, as above)

$$\Delta^{\mathbb{R}^n}(|x|^{-s}F) = -s|x|^{-(s+2)}((-s+n-2)F + 2EF) + |x|^{-s}\Delta^{\mathbb{R}^n}F$$

Since F is positive homogeneous of degree s, we see $EF = sF$. Thus, recalling $\lambda = -s(s+n-2)$ and that and that $F(x) = |x|^s g(x)$, we have

$$|x|^{-2}\Delta^{\mathbb{R}^n}g(x) = \lambda g(x) + |x|^{-s+2}\Delta^{\mathbb{R}^n}F(x) \quad \text{for all } x \in \mathbb{R}^n - \{0\}.$$

Now since g is positive homogeneous of degree zero, $\Delta^{\mathbb{R}^n}g$ is positive homogeneous of degree -2, so

$$|x|^{-2}\Delta^{\mathbb{R}^n}g(x) = (\Delta^{\mathbb{R}^n}g)(x/|x|) = \Delta^S g(x/|x|) = \lambda g(x).$$

Thus, plugging in to the penultimate display

$$\lambda g(x) = \lambda g(x) + |x|^{-s+2}\Delta^{\mathbb{R}^n}F(x) \quad \text{for all } x \in \mathbb{R}^n - \{0\}.$$

Canceling, we find

$$|x|^{-s+2}\Delta^{\mathbb{R}^n}F(x) = 0 \quad \text{for all } x \in \mathbb{R}^n - \{0\}$$

giving $\Delta^{\mathbb{R}^n}F = 0$ as claimed.

Recapitulating, we have just shown

> The eigenfunctions of Δ^S are precisely *harmonic, homogeneous* functions on $\mathbb{R}^n$, restricted to S.

Harmonic homogeneous functions are polynomials

The Fourier transform of a smooth, rapidly decaying function $\varphi : \mathbb{R}^n \to \mathbb{C}$ is a function $\mathcal{F}\varphi : \mathbb{R}^n \to \mathbb{C}$ defined by

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^n} \varphi(x)e^{-i2\pi\langle\xi,x\rangle}\,\mathrm{d}x.$$

The salient feature here, of the Fourier transform is that it converts differentiation to multiplication; integrating by parts, compute

$$\mathcal{F}(\frac{\partial}{\partial x_j}\varphi)(\xi) = \int_{\mathbb{R}^n} \varphi_j(x)e^{-2\pi i\langle\xi,x\rangle}\,\mathrm{d}x = -i2\pi\xi_j\int_{\mathbb{R}^n}\varphi(x)e^{-i\langle\xi,x\rangle}\,\mathrm{d}x = -i2\pi\xi_j\mathcal{F}\varphi(\xi).$$

In particular,

$$\mathcal{F}(\Delta^{\mathbb{R}^n}\varphi)(\xi) = -|2\pi\xi|^2\mathcal{F}\varphi(\xi).$$

If φ is harmonic, then the last display shows

$$|\xi|^2\mathcal{F}\varphi(\xi) = 0 \quad \text{for all } \xi \in \mathbb{R}^n.$$

Since $|\xi|^2$ is supported away from zero, the smooth function $\mathcal{F}\varphi$ must have support concentrated at zero, which is impossible unless $\mathcal{F}\varphi$ is identically zero, which would require that φ itself is identically zero.

We know, however, that there *are* nonzero harmonic functions; for example, harmonic polynomials. The existence of such functions imposes no paradox, since we assumed that φ *decayed rapidly* in addition to being smooth, which apparently does not hold for nonzero harmonic functions. Nonetheless, the computation that

$$\Delta^{\mathbb{R}^n}\varphi = 0 \quad \text{implies} \quad |\xi|^2\mathcal{F}\varphi(\xi) = 0$$

suggests some usefulness in enlarging the domain of the Fourier transform to include, at the very least, harmonic positive homogeneous functions.

We have already shown that eigenfunctions of Δ^S come from positive homogeneous functions on $\mathbb{R}^n$. For an eigenfunction $f : S \to \mathbb{C}$, let $F(x) = |x|^s f(x/|x|)$ be the harmonic, degree s homogeneous extension discussed in the last section. For $x \in \mathbb{R}^n - \{0\}$, there is some constant C giving the uniform bound $|f(x/|x|)| \le C$, since S is compact. By positive homogeneity, the bound on the sphere demonstrates *moderate growth*,

$$|F(x)| \le C|x|^s \quad \text{for all } x \in \mathbb{R}^n - \{0\}.$$

Furthermore as $x \to 0$, the display shows that $F(x) \to 0$. Thus, F is locally integrable with moderate growth on all of $\mathbb{R}^n$, so it defines a *tempered* distribution

$$u_F : \varphi \longmapsto \int_{\mathbb{R}^n} \varphi(x)F(x)\,\mathrm{d}x.$$

The Fourier transform naturally extends to tempered distributions, viz

$$\mathcal{F}u_F(\varphi) = u_F(\mathcal{F}\varphi).$$

As a literal function, F is harmonic, so integration by parts shows that u_F is harmonic *as a distribution*,

$$\Delta^{\mathbb{R}^n}u_F(\varphi) = u_F(\Delta^{\mathbb{R}^n}\varphi) = u_{\Delta^{\mathbb{R}^n}F}(\varphi) = 0,$$

for all smooth, rapidly decaying φ.

Compute for all smooth, rapidly decaying φ,

$$0 = \mathcal{F}\Delta^{\mathbb{R}^n}u_F(\varphi) = \mathcal{F}u_F(\Delta^{\mathbb{R}^n}\varphi) = u_F(-|\xi|^2\mathcal{F}\varphi) = (-|\xi|^2 \cdot \mathcal{F}u_F)(\varphi).$$

Thus, we have the distributional equality

$$-|\xi|^2 \mathcal{F} u_F = 0.$$

Since $-|\xi|^2$ is supported away from 0, this display demonstrates that u_F has support concentrated at 0.

As proved [10], the only tempered distributions supported at $\{0\}$ are *finite* linear combinations of the Dirac delta distribution $\delta : \varphi \longmapsto \varphi(0)$, and its (distributional) derivatives. Granting this, we see

$$\mathcal{F} u_F = \sum_\alpha c_\alpha \delta^\alpha \qquad \text{finitely many multi-indices } \alpha = (\alpha_1, .., \alpha_d)\ .$$

Since the (inverse) Fourier transform of δ (and its derivatives) are *polynomials*, we see

$$u_F = \mathcal{F}^{-1} \mathcal{F} u_F = \mathcal{F}^{-1}(\sum_\alpha c_\alpha \delta^\alpha) = u_{\sum_\alpha c_\alpha x^\alpha}.$$

This equality of tempered distributions implies an equality of literal *functions*

$$F = \sum_\alpha c_\alpha x^\alpha.$$

That is, F is a harmonic positive homogeneous *polynomial*. Furthermore, since the homogeneity degree of a polynomial must be an integer, we find that the eigenvalues of Δ^S are of the form $\lambda_d = -d(d+n-2) = -(d+\frac{n-1}{2})^2 + (\frac{n-1}{2})^2$ for $d \in \mathbb{Z}_{\geq 0}$. In particular, we see that Δ^S distinguishes harmonic homogeneous polynomials of distinct degree, so distinct degree (harmonic homogeneous) polynomials are *orthogonal* in $L^2(S)$.

Conclusion

We have shown that the differential equation

$$\Delta^S f = \lambda f \qquad \text{for } f \in C^\infty(S) \text{ and } \lambda \in \mathbb{R}_{\leq 0}$$

has nonzero solutions if and only if $\lambda = \lambda_d = -(d + \frac{n-1}{2})^2 + (\frac{n-1}{2})^2$ for $d \in \mathbb{Z}_{\geq 0}$. Further, we have determined that the λ_dth eigenspace is precisely

$$\mathfrak{H}^d = \text{Harmonic homogeneous polynomials of degree } d \text{ restricted to } S.$$

Thus,

With proj_d denoting the projection of $L^2(S)$ onto $\mathfrak{H}^d$, we have

$$L^2(S) = \bigoplus_{d \in \mathbb{Z}_{\geq 0}} \mathfrak{H}^d \quad \text{with} \quad f = \sum_{d \in \mathbb{Z}_{\geq 0}} \text{proj}_d\, f$$

3.3 Retrospective discovery of a representation

A problem

In the last section, we showed that $L^2(S)$ decomposes into (the restriction to the sphere of) harmonic, homogeneous polynomials. In particular, *any* homogeneous polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]^d$ restricted to the sphere agrees with a finite sum of *harmonic, homogeneous* polynomials, restricted to the sphere: with each f_i *harmonic, homogeneous* of degree i,

$$f\big|_S = f_d\big|_S + f_{d-2}\big|_S + \ldots$$

On the other hand as a homogeneous function on $\mathbb{R}^n$,

$$f = f_d + r^2 f_{d-2} + \ldots$$

Evidently, the direct sum in the grading

$$\mathbb{C}[x_1, \ldots, x_n] = \bigoplus_{d \geq 0} \mathbb{C}[x_1, \ldots, x_n]^d$$

collapses upon restriction to the sphere: any two polynomials which differ by a polynomial multiple of $1 - r^2$ will agree upon restricting.

In order for an operator D on $\mathbb{C}[x_1, \ldots, x_n]$ to act on the sphere, it must be constant on all 'lifts' by r^2. A sufficient condition for D to act on the sphere is if $D(r^2 f) = r^2 Df$, since then $D(r^2 f)\big|_S = Df\big|_S$. By construction, the spherical Laplacian Δ^S is one such operator. That said, we have the formula

$$\Delta^S f\big|_S = (-E^2 f - (2-n)Ef + \Delta^{\mathbb{R}^n} f)\big|_S \quad \text{for } f \in \mathbb{C}[x_1, \ldots, x_n]$$

Strangely enough, the operators $E = \sum_i x_i \frac{\partial}{\partial x_i}$ and $\Delta^{\mathbb{R}^n} = \sum_i \frac{\partial^2}{\partial x_i^2}$, which make up Δ^S are *not* constant on lifts by r^2. Rather, compute

$$\Delta^{\mathbb{R}^n}(r^2 f) = (\Delta^{\mathbb{R}^n} r^2) f + 2\sum_{i=1}^{n} x_i f_i + r^2 \Delta^{\mathbb{R}^n} f = (2n + 4E)f + r^2 \Delta^{\mathbb{R}^n} f,$$

and

$$E(r^2 f) = E(r^2) f + r^2 Ef = 2r^2 f + r^2 Ef.$$

Since the eigenvalues of E on $\mathbb{C}[x_1, \ldots, x_n]$ are *positive*, $(2n+4E)f$ is never zero so $\Delta^{\mathbb{R}^n}(r^2 f)$ is never $r^2 \Delta^{\mathbb{R}^n} f$. Further, since no *function* is annihilated by r^2, we see $E(r^2 f)$ is never $r^2 Ef$.

While E and $\Delta^{\mathbb{R}^n}$ *fail* to act on the sphere, the fail in a very interesting way. The computations have just shown that as operators on $\mathbb{C}[x_1, \ldots, x_n]$,

$$[\Delta^{\mathbb{R}^n}, r^2] = 4(\frac{n}{2} + E) \quad \text{and} \quad [E, r^2] = 2r^2.$$

The failure of these operators to commute mimics (up to some normalizations) some of the bracket relations in the Lie algebra

$$\begin{aligned}\mathfrak{sl}_2 &= \text{ Lie algebra of } \mathrm{SL}_2(\mathbb{R}) \\ &= \text{ 3 dimensional real vectorspace of matrices with trace 0.}\end{aligned}$$

with basis

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

satisfying

$$[x, y] = h \quad [h, x] = 2x \quad [h, y] = -2y.$$

Inspired by this realization, hoping for a relation between E and $\Delta^{\mathbb{R}^n}$, compute

$$\begin{aligned}[E, \Delta^{\mathbb{R}^n}] &= E(\Delta^{\mathbb{R}^n} f) - \Delta^{\mathbb{R}^n}(Ef) \\ &= \sum_{i=1}^{n} (Ef_{ii} - \Delta^{\mathbb{R}^n}(x_i f_i)) \\ &= \sum_{i=1}^{n} (Ef_{ii} - (\Delta^{\mathbb{R}^n} x_i) f_i - 2(\sum_{j=1}^{n} f_{ij}) - x_i(\Delta^{\mathbb{R}^n} f_i)) \\ &= -\sum_{i=1}^{n} 2 f_{ii} = -2\Delta^{\mathbb{R}^n} f.\end{aligned}$$

Renormalizing, we have just discovered that

$$\begin{aligned} x &\longmapsto \text{ multiplication by } r^2/2 \\ y &\longmapsto \text{ application of } -\Delta^{\mathbb{R}^n}/2 \\ h &\longmapsto \text{ application of } n/2 + E, \end{aligned}$$

is a *representation* of $\mathfrak{sl}_2$ on $\mathbb{C}[x_1, \ldots, x_n]$. This representation is called the Segal–Shale **oscillator representation** of $\mathfrak{sl}_2$. This representation is discussed in [22].

Irreducible subrepresentations in $\mathbb{C}[x_1, \ldots, x_n]$: lowest weight theory

We close this section by identifying the irreducible subrepresentations of the oscillator representation on $\mathbb{C}[x_1, \ldots, x_n]$. My intention is twofold:

- The decomposition of $\mathbb{C}[x_1, \ldots, x_n]$ into $\mathfrak{sl}_2$-invariant subspaces gives a decisive explanation for the collapse in gradation of $\mathbb{C}[x_1, \ldots, x_n]$ upon restriction to the sphere, providing a response to the concerns in the opening of this section.

- The techniques serve as an archetype for computations in the representation of $\mathfrak{sl}_2$, and will be indispensable in our analysis of $L^2(\mathbb{R}^n)$.

Since what follows applies to general representations of $\mathfrak{sl}_2$, we change some notation. Let $\mathfrak{g} = \mathfrak{sl}_2$, let $V = \mathbb{C}[x_1, \ldots, x_n]$, and let $v \in V$. The representation $\mathfrak{g}v$ **generated** by v is the complex vectorspace spanned by $\mathfrak{g}$ translates of v. Using the commutation relations inductively[3], we can write such a representation in the form[4]

$$\mathfrak{g}v = \sum_{a,b,c \geq 0} \mathbb{C}x^a y^b h^c v.$$

With no stipulation on $v \in V$, there is not much one can say about $\mathfrak{g}v$.

Suppose that v is an eigenvector of h, with eigenvalue λ. From the relation $[h, x] = 2x$, compute

$$hxv = ([h, x] + xh)v = (2x + xh)v = x(2 + h)v = (\lambda + 2)xv.$$

That is, xv is again an eigenvector of h, but now with eigenvalue $\lambda + 2$. Similarly, from the relation $[h, y] = -2y$, compute

$$hyv = ([h, y] + yh)v = (-2y + yh)v = y(-2 + h)v = (\lambda - 2)yv,$$

so yv is an eigenvector of h, with eigenvalue $\lambda - 2$.

It is standard to call an eigenvector of h a **weightvector**. The eigenvalue of a weightvector is its **weight**. The representation generated by a weightvector is called a **weightspace**. We have just shown that x *raises* the weight of a weightvector by 2, just as y lowers the weight by 2. Correspondingly, we call x the **raising operator** and y the **lowering operator**. Many books call x and y *creation* and *annihilation* operators, respectively. We reserve that terminology for other operators, to appear later.

For a weightvector v with weight λ, the vectors xyv and yxv are again weightvectors with weight λ, but in general there is no reason to believe that they are identical. That is, *weightspaces need not be one dimensional*, and further *xy and yx need not act by a scalar*. Nonetheless, the representation generated by a weightvector can be slightly simplified

$$\mathfrak{g}v = \bigoplus_{a,b \geq 0} x^a y^b v = \bigoplus_{a,b \geq 0} (\lambda + 2a - 2b \text{ weightspace}),$$

though each summand need not be stable under the action of $\mathfrak{g}$.

Remark 4. In our specific scenario, where $h = n/2 + E$, we know that the eigenspaces of E in $V = \mathbb{C}[x_1, \ldots, x_n]$ are the spaces of homogeneous polynomials $\mathbb{C}[x_1, \ldots, x_n]^d$. That is, the $n/2 + d$ weightspace in V is $\mathbb{C}[x_1, \ldots, x_n]^d$. We have just argued that h acts on $r^2\mathbb{C}[x_1, \ldots, x_n]^d \subset \mathbb{C}[x_1, \ldots, x_n]^{d+2}$ by $n/2 + d + 2$, as anticipated. Likewise, h acts on $\Delta^{\mathbb{R}^n}\mathbb{C}[x_1, \ldots, x_n]^d \subset \mathbb{C}[x_1, \ldots, x_n]^{d-2}$ by $n/2 + d - 2$. The claim that $xy = -1/4r^2\Delta^{\mathbb{R}^n}$

[3] The process of using commutation relations inductively at the expense of picking up lower degree terms, proves the easy part of the Poincare–Birkoff–Witt theorem.

[4] The notation $\mathfrak{g}v$ in this display is slightly dishonest. Lie algebras do not equipped with a notion of composition, outside of the bracket $[a, b]$. Thus, $\mathfrak{g}$ gives no indication of the behavior of entities such as $xyhv$. The suitable environment to work with composition of Lie algebra elements is the *universal enveloping algebra*, the smallest unital associative algebra containing $\mathfrak{g}$, such that the Lie bracket is given by the commutant.

and $yx = -1/4\Delta^{\mathbb{R}^n} r^2$ do not agree on $\mathbb{C}[x_1, \ldots, x_n]^d$ was verified earlier. Furthermore, the claim that '$\mathbb{C}[x_1, \ldots, x_n]$ is graded by degree' can be reinterpreted as a decomposition

$$\mathbb{C}[x_1, \ldots, x_n] = \bigoplus_{d \geq 0} \mathbb{C}[x_1, \ldots, x_n]^d = \bigoplus_{d \geq 0} ((n/2 + d) \text{ weightspace}).$$

This decomposition is suboptimal from the point of view of representation theory, because the summands are not stable under the whole action of $\mathfrak{sl}_2$. In what follows, we do decompose $\mathbb{C}[x_1, \ldots, x_n]$ into a sum of subrepresentations.

From the bracket relation $[x, y] = h$ compute

$$yxh = ([x, y] - xy)v = hv - xyv.$$

Thus, in the special circumstance that y *annihilates* a weightvector $yv = 0$, we find

$$yxh = hv - xyv = \lambda v + 0 = \lambda v,$$

showing that yx *does* act as a scalar on v. In this case the representation generated by v simplifies further,

$$\mathfrak{g}v = \bigoplus_{a \geq 0} \mathbb{C}x^a v.$$

The sum is direct because each of $v, xv, x^2 v, \ldots$ have distinct h-eigenvalues.

We call a y-annihilated weightvector a **lowest** weightvector, and the representation generated by v is a **lowest weightspace**.

Remark 5. The vectors annihilated by $y = -\Delta^{\mathbb{R}^n}/2$ are the *harmonic polynomials*. The lowest weightvectors in $\mathbb{C}[x_1, \ldots, x_n]$ are the *harmonic, homogeneous polynomials*. For $f \in \mathfrak{H}^d$, harmonic and homogeneous of degree d, a $n/2 + d$ lowest weightspace is

$$\mathfrak{g}f = \bigoplus_{a \geq 0} \mathbb{C}r^{2a} f = \mathbb{C} \cdot f \oplus \mathbb{C} \cdot r^2 f \oplus \mathbb{C} \cdot r^4 f \oplus \ldots$$

The sum is direct since the summands are in distinct eigenspaces for h. Note that for each a, the lowest weightspace $\mathfrak{g}f$ has one dimensional intersection with $\mathbb{C}[x_1, \ldots, x_n]^{d+2a}$. Furthermore, since

$$-\frac{1}{4}\Delta^{\mathbb{R}^n}(r^2 f) = yxf = ([y, x] + xy)f = (-h + xy)f = -hf = -(n/2 + d)f,$$

we find that, by induction, any nonconstant polynomial in that intersection is *not* harmonic. One might suspect, and indeed it is the case, that the lowest weightspaces 'fill up' all of $\mathbb{C}[x_1, \ldots, x_n]$. That is,

$$\mathbb{C}[x_1, \ldots, x_n] = \bigoplus_{d \geq 0} (n/2 + d) \text{ lowest weightspaces}$$

This is proved next, and relies on three general phenomena:

- Lowest weightspaces are irreducible
- For any weight λ, there is just one λ-*lowest weightspace* M_λ, up to $\mathfrak{sl}_2$-isomorphism.
- For a weightvector v and weight λ such that $y^m v = 0$, for some m, the weightspace $\mathfrak{g}v$ is generated by a *finite* direct sum of *lowest* weightspaces, each with lowest weight in $\{\lambda, \lambda-2, ..., \lambda-2m\}$ occurring at most once.

Irreducibility of lowest weightspaces To show irreducibility, note that any nonzero subrepresentation of $\mathfrak{g}v$ will contain a vector of the form $c_0 v + c_1 xv + c_2 x^2 v + ... + c_d x^d v$ with $c_0 \neq 0$, by applying y sufficiently many times. Each $x^j v$ is an eigenvector of h with eigenvalue $\lambda+2j$, so is annihilated by $h-(\lambda+2j)$. On the other weightvectors, this operator acts by a nonzero scalar. Doing this for each j up to d shows that such a subrepresentation would contain a multiple of the generator v. Thus, that subrepresentation is all of $\mathfrak{g}v$.

Uniqueness of lowest weightspaces Granting irreducibility, uniqueness follows easily. Indeed, let v and w be two nonzero lowest weightvectors, both with weight λ. Then the lowest weightspaces are

$$\mathfrak{g}v = \bigoplus_{d\geq 0} \mathbb{C}x^d v \quad \text{and} \quad \mathfrak{g}w = \bigoplus_{d\geq 0} \mathbb{C}x^d w.$$

Define a map $T : \mathfrak{g}v \longrightarrow \mathfrak{g}w$ by $x^a v \longmapsto x^a w$. The map T is equivariant under x, by construction. Since v and w share the weight λ, T is equivariant under h. Last, compute

$$\begin{aligned} T(yx^a v) &= T((\lambda-2a)x^{a-1}v) \\ &= (\lambda-2a)T(x^{a-1}v) \\ &= (\lambda-2a)x^a w = yx^a w = yT(x^a v). \end{aligned}$$

Thus T is a $\mathfrak{g}$-map. Since $\mathfrak{g}v$ and $\mathfrak{g}w$ are irreducible, T is an isomorphism by Schur's lemma. Conversely, for any $\mathfrak{g}$-map T between lowest weight representations $\mathfrak{g}v$, with weight λ and $\mathfrak{g}w$ with weight μ, we need $T(hv) = T(\lambda v) = \lambda T(v)$ by linearity. By equivariance, we also need $T(hv) = hT(v)$, so h must act on $T(v)$ by λ. On the other hand, by linearity $T(yv) = T(0) = 0$ and by equivariance, $T(yv) = yT(v)$, so $yT(v) = 0$. That is, $T(v)$ is a lowest weightvector, with weight λ, as claimed.

We let M_λ denote the unique λ-lowest weightspace up to $\mathfrak{g}$-isomorphism.

Finite generation of eventually annihilated weightspaces Let v be a weightvector with weight λ, and suppose $y^{m+1}v = 0$ with $m+1$ minimal. We prove by induction on m that $\mathfrak{g}v$ is a *finite* direct product of *lowest* weightspaces with weights in $\{\lambda, \lambda-2, ..., \lambda-2m\}$ occurring at most once. For $m = 0$, then $yv = 0$ and $\mathfrak{g}v$ is a lowest weightspace, by definition.

Let $m > 0$. We want to find some weightvector $v' \in \mathfrak{g}v$ with $y^m v' = 0$, since then by induction $\mathfrak{g}y^m v$ is a finite direct sum of lowest weightspaces with weights in $\{\lambda, ..., \lambda -$

$2(m-1)\}$ occurring at most once. We look for v' as an element of the form $cv + x^m y^m v$ for constant c. Suppose $y^m x^m y^m v \neq 0$. By the assumption on v, we see $y(y^m x^m y^m v) = y^{m+1} x^m y^m v = 0$, so $y^m x^m y^m v$ is in the irreducible $\lambda - 2m$ lowest weightspace generated by $y^m v$. Further, since $y^m x^m y^m v$ has weight $\lambda - 2m$, it is a multiple of the generator: $y^m x^m y^m v = c y^m v$. Next, observe that $c \neq 0$: if $y^m x^m y^m v = 0$, then for some $j < m$ we'd have $y^j x^m y^m v$ being a $\lambda - 2j$ lowest weightvector, which would generate a proper subrepresentation in $\mathfrak{g} y^m v$, contradicting irreducibility. Thus $c \neq 0$ and $v' = x^m y^m v - cv$ is annihilated by y^m, and consequently by induction hypothesis decomposes into a finite direct sum of lowest weightspaces with weights in $\{\lambda, ..., \lambda - (2m-1)\}$. Last, since $y^m v$ is a lowest weightvector with weight $\lambda - 2m$ we see

$$\mathfrak{g} v = M_{\lambda - 2m} + \bigoplus M_\mu \quad \text{with } \mu \in \{\lambda, ..., \lambda - 2(m-1)\},$$

and we want to show that the sum is direct. If $M_{\lambda-2m}$ has nonzero intersection with the direct sum, then it must have nonzero intersection with one of the summands M_μ. Since both M_μ and $M_{\lambda-2m}$ are irreducible, we conclude $M_{\lambda-2m} = M_\mu$, contradicting the assumption that $\mu \in \{\lambda, ..., \lambda - 2(m-1)\}$. Thus the sum is direct, completing the induction.

Now, returning to the situation at hand, start with the decomposition of $V = \mathbb{C}[x_1, \ldots, x_n]$ into weightspaces, induced by the degree-grading

$$\mathbb{C}[x_1, \ldots, x_n] = \bigoplus_{d \geq 0} \mathbb{C}[x_1, \ldots, x_n]^d = \bigoplus_{d} (n/2 + d) \text{ weightspace.}$$

The Euclidean Laplacian decreases the degree of homogeneity (i.e. weight) of a polynomial by 2, so for $f \in \mathbb{C}[x_1, \ldots, x_n]^d$, we have $y^{\lceil d/2 \rceil} f \in \mathbb{C}[x_1, \ldots, x_n]^0 = \mathbb{C} \cdot 1$. Thus $y^{\lceil d/2 \rceil + 1} f = 0$, and so the representation generated by f is a finite direct sum of representations generated by *harmonic* homogeneous polynomials with degrees in $\{d, d-2, ...0\}$ each occurring at most once. That is, for some harmonic homogeneous f_i of degree $i \in \{d, d-2, ...0\}$,

$$\mathfrak{sl}_2 f = \mathfrak{sl}_2 f_d \oplus \mathfrak{sl}_2 f_{d-2} \oplus \cdots$$

But since each f_i is harmonic and homogeneous (a lowest weightvector), the summands are

$$\mathfrak{sl}_2 f_i = \mathbb{C} f_i \oplus \mathbb{C} r^2 f_i \oplus \mathbb{C} r^4 f_i \cdots .$$

Thus, letting f run through a basis of $\mathfrak{H}^a$

$$\mathbb{C}[x_1, \ldots, x_n]^d \subset \bigoplus_{a=0}^{d} (\bigoplus_{f} \mathfrak{sl}_2 f) \approx \bigoplus_{a=0}^{d} \dim_{\mathbb{C}}(\mathfrak{H}^a) \cdot M_{n/2+a}.$$

Consequently, with each summand stable under the action of $\mathfrak{sl}_2$,

$$\mathbb{C}[x_1, \ldots, x_n] \approx \bigoplus_{d \geq 0} \dim_{\mathbb{C}}(\mathfrak{H}^d) \cdot M_{n/2+d} \quad \text{as } \mathfrak{sl}_2\text{-spaces.}$$

Reenabling the action of $\mathrm{SO}(n)$ In the decomposition displayed above, the oscillator representation cannot distinguish between the $\dim_{\mathbb{C}}(\mathfrak{H}^d)$ copies of $M_{n/2+d}$. We can remove the multiplicity, at the expense of enlarging the irreducibles by reenabling the action of $\mathrm{SO}(n)$.

Since each of the operators r^2, $\Delta^{\mathbb{R}^n}$, and E commute with the action of $\mathrm{SO}(n)$, we may view $\mathbb{C}[x_1,\ldots,x_n]$ as a representation of $\mathrm{SO}(n)\times\mathfrak{sl}_2$. Explicitly, for $(g, ax+by+ch)\in\mathrm{SO}(n)\times\mathfrak{sl}_2$ and $f(x)\in\mathbb{C}[x_1,\ldots,x_n]$, define

$$(g, ax+by+ch)\times f(x)\longmapsto (ax+by+ch)f(xg).$$

Now fix a nonzero harmonic homogeneous polynomial f of degree d. We have seen that the $\mathfrak{sl}_2$ representation generated by f is of the form

$$\mathfrak{sl}_2 f=\mathbb{C}f\oplus\mathbb{C}r^2f\oplus\mathbb{C}r^4f\oplus\cdots\approx M_{n/2+d}.$$

Then, since the action of $\mathrm{SO}(n)$ is transitive on $\mathfrak{H}^d$,

$$(\mathrm{SO}(n)\times\mathfrak{sl}_2)f=\mathfrak{H}^d\oplus r^2\mathfrak{H}^d\oplus r^4\mathfrak{H}^d\oplus\cdots.$$

Now, since each of the summands $r^{2a}\mathfrak{H}^d$ is stable under the action of $\mathrm{SO}(n)$, we have just shown

$$(\mathrm{SO}(n)\times\mathfrak{sl}_2)f\approx\mathfrak{H}^d\otimes M_{n/2+d}.$$

Thus, the presence of the multiplicity $\dim_{\mathbb{C}}(\mathfrak{H}^d)$ of $M_{n/2+d}$ in the decomposition of $\mathbb{C}[x_1,\ldots,x_n]$ as an $\mathfrak{sl}_2$-*subrepresentation* is replaced by *the unique* $\mathrm{SO}(n)\times\mathfrak{sl}_2$-*subrepresentation* $\mathfrak{H}^d\otimes M_{n/2+d}$. Thus, we have a *multiplicity free* decomposition into irreducible subrepresentations

$$\mathbb{C}[x_1,\ldots,x_n]\approx\bigoplus_{d\geq 0}\mathfrak{H}^d\otimes M_{n/2+d}\quad(\text{as }\mathrm{SO}(n)\times\mathfrak{sl}_2\text{ spaces}).$$

Explanation of the collapse

As a space of functions, polynomials restricted to the sphere may be identified with the coordinate ring of the variety $\{x\in\mathbb{R}^n : |x|^2=r^2=1\}$. That is,

$$\mathbb{C}[x_1,\ldots,x_n]\big|_S\approx\mathbb{C}[x_1,\ldots,x_n]/(r^2-1)\quad(\text{as function spaces}).$$

It follows that an operator will make sense on $\mathbb{C}[x_1,\ldots,x_n]\big|_S$ exactly when it preserves the ideal $(r^2-1)\subset\mathbb{C}[x_1,\ldots,x_n]$. Observe that $h(r^2-1)=2r^2+n/2$ is *never* a polynomial multiple of r^2-1. Similarly, $\Delta^{\mathbb{R}^n}(r^2-1)=2$. Thus, the oscillator representation on $\mathbb{C}[x_1,\ldots,x_n]$ emphatically does *not* restrict to a representation on $\mathbb{C}[x_1,\ldots,x_n]\big|_S$.

This failure to restrict is consonant with our depiction of the lowest weightspace, given that the *infinite* dimensional representations $\mathfrak{sl}_2 f$ (for $f\in\mathfrak{H}^d$) collapses to a *one* dimensional vectorspace

$$(\mathfrak{sl}_2 f)\big|_S=\mathbb{C}f\quad(\text{as function spaces, no longer representations}).$$

On the other hand, $\mathrm{SO}(n)$ *does* preserve the relation $r^2 - 1$. Further, since $\Delta^{\mathbb{R}^n}$ is $\mathrm{SO}(n)$-invariant, the decomposition as an $\mathrm{SO}(n)$ representation *does* restrict to the sphere

$$\mathbb{C}[x_1, \dots, x_n]\big|_S = \bigoplus_{d \geq 0} \mathfrak{H}^d \quad \text{(as SO}(n)\text{ representations)}$$

Thus the failure of E and $\Delta^{\mathbb{R}^n}$ to commute with r^2, in turn resulting in the failure of the oscillator representation to restrict to the sphere, can be traced to the oscillator representation acting nontrivially on $M_{n/2+d}$. That is, the oscillator representation stores information about structure *too fine* in $\mathbb{C}[x_1, \dots, x_n]$ to restrict to the *coarser* object $\mathbb{C}[x_1, \dots, x_n]/(r^2 - 1)$.

However, the Casimir operator *now coming from* $\mathfrak{sl}_2$ is (in part) *designed* to act as a scalar on irreducible representations of $\mathfrak{sl}_2$. That is, while the oscillator representation of $\mathfrak{sl}_2$ does not restrict to representation on $\mathbb{C}[x_1, \dots, x_n]\big|_S$

the image of the Casimir element will.

We compute the image as follows. With respect to the nondegenerate bilinear pairing $\langle A, B \rangle = \mathrm{tr}(AB^\top)$ a dual basis for $\mathfrak{sl}_2$ is

$$h^* = \frac{1}{2}h \quad x^* = y \quad y^* = x.$$

The Casimir element is then, using the relation $yx = [y, x] + xy = -h + xy$ in the second equality,

$$\Omega^{\mathfrak{sl}_2} = \frac{1}{2}h^2 + xy + yx = \frac{1}{2}h^2 - h + 2xy.$$

Identifying $\Omega^{\mathfrak{sl}_2}$ with its image under the oscillator representation, this shows that the Casimir element acts on the sphere. On the other hand, compute

$$\begin{aligned}
\Omega^{\mathfrak{sl}_2} &= \frac{1}{2}(\frac{n}{2} + E)^2 - (\frac{n}{2} + E) - 2\frac{1}{4}r^2\Delta^{\mathbb{R}^n} \\
&= \frac{1}{2}(\frac{n^2}{4} + nE + E^2) - \frac{n}{2} - E - \frac{1}{2}r^2\Delta^{\mathbb{R}^n} \\
&= \frac{1}{2}E^2 - \frac{2-n}{2}E - \frac{1}{2}r^2\Delta^{\mathbb{R}^n} + (\frac{n^2}{8} - \frac{n}{2}).
\end{aligned}$$

Recalling the formula

$$\Delta^{\mathbb{R}^n} = -E^2 - (2-n)E - r^2\Delta^{\mathbb{R}^n}$$

we have just discovered that

$$2\Omega^{\mathfrak{sl}_2} = \Delta^S + n(\frac{n}{4} - 1).$$

That is up to normalization, the image of the Casimir element *for* $\mathfrak{sl}_2$ *is* the spherical Laplacian. The image of the Casimir element for $\mathfrak{sl}_2$ under the oscillator representation ignores *just enough* differential structure on $\mathbb{C}[x_1, \dots, x_n]$ to reproduce our results on spherical harmonics.

3.4 Discrete decomposition of $L^2(\mathbb{R}^n)$

3.4.1 First pass: Spectral theory of $L^2(\mathbb{R}^n)$, immediate catastrophe

Unlike the torus $\mathbb{T}^n \approx \mathbb{R}^n/\mathbb{Z}^n$ and the sphere $S \approx \mathrm{SO}(n-1) \setminus \mathrm{SO}(n)$, n-dimensional Euclidean space is *not* compact. Nonetheless, $\mathbb{R}^n$ is a locally compact abelian Lie group, so it has an invariant measure $\mathrm{d}\mu$ and invariant (Casimir) operator $\Omega^{\mathbb{R}^n} = \Delta^{\mathbb{R}^n} = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. However, the norms $|\cdot|^2_{L^2(\mathbb{R}^n)} : f \longmapsto \int_{\mathbb{R}^n} |f|^2 \, \mathrm{d}\mu$ and $|\cdot|^2_{H^1(\mathbb{R}^n)} : f \longmapsto |f|^2_{L^2(\mathbb{R}^n)} + |f'|^2_{L^2(\mathbb{R}^n)}$ are not sensible on the whole complex vectorspace

$$C^\infty(\mathbb{R}^n) = \text{ smooth functions } f : \mathbb{R}^n \to \mathbb{C},$$

since smoothness imposes no constraint on the decay of a function.

Even if we revert to the classical framework, and work backwards by defining (with D^1 the space of once differentiable functions, and a.e. abbreviating almost everywhere)

$$\begin{aligned} L^2(\mathbb{R}^n) &= \text{measurable } f : \mathbb{R}^n \to \mathbb{C} \text{ with } |f|_{L^2(\mathbb{R}^n)} < \infty \\ H^1(\mathbb{R}^n) &= D^1 \text{ a.e. } f : \mathbb{R}^n \to \mathbb{C} \text{ such that } f' \in L^2(\mathbb{R}^n) \text{ and } |f|_{H^1(\mathbb{R}^n)} < \infty \\ C_0^\infty(\mathbb{R}^n) &= L^2(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n) \end{aligned}$$

then the embedding $H^1(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is *not compact*[5]. Thus, we cannot apply the spectral theorem to obtain an orthonormal basis of $\Delta^{\mathbb{R}^n}$ eigenfunctions.

In any case, the differential equation

$$\Delta^{\mathbb{R}^n} f = \lambda f \qquad \text{for } \lambda \in \mathbb{R}_{\leq 0}$$

is easily solved, giving $\lambda = -|\xi|^2$ corresponding to the $(\xi_1, ..., \xi_n)$-frequency complex exponential $f(x) = e^{i\langle x, \xi \rangle}$. That none of these solutions are square integrable corroborates our observation that the inclusion $H^1(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n)$ is not compact.

Evidently, we need to find a different operator than $\Delta^{\mathbb{R}^n}$. We will find it in the oscillator representation, as follows.

The oscillator representation afforded a decomposition of $\mathbb{C}[x_1, \ldots, x_n]$, a space of functions on $\mathbb{R}^n$, into irreducible subrepresentations of $\mathrm{SO}(n) \times \mathfrak{sl}_2$. This decomposition does not interact well with $L^2(\mathbb{R}^n)$ because polynomials do not decay. Furthermore, all weightvectors of the operator $h = E + n/2$, polynomial or not, are positive *homogeneous*, and thus grow like polynomials.

In order to find such an operator, we will suppose momentarily that the oscillator representation comes from some representation of the underlying Lie group, $\mathrm{SL}_2(\mathbb{R})$ on $L^2(\mathbb{R}^n)$.

[5] Recall that compactness of the map $H^1(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is equivalent to the image of every bounded sequence in $H^1(\mathbb{R}^n)$ having a convergent subsequence in $L^2(\mathbb{R}^n)$. Consider a smooth function ψ supported on the unit ball about the origin, of unit $H^1(\mathbb{R}^n)$ norm. Define a sequence of functions $\psi_i(x_1, .., x_i) = \psi(x_1 + i, ..., x_n + i)$. Since $\mathrm{d}\mu$ and $\Delta^{\mathbb{R}^n}$ are translation invariant, each ψ_i has unit $H^1(\mathbb{R}^n)$ norm. The supports of the ψ_i are pairwise disjoint, and thus orthogonal in $L^2(\mathbb{R}^n)$, so cannot be Cauchy.

Supposing such a representation were *unitary*, meaning

$$\langle gf, g\varphi\rangle = \langle f, \varphi\rangle \quad (\text{for all } g \in \mathrm{SL}_2(\mathbb{R}), \text{ and } f, \varphi \in C^\infty(R^n))$$

then compute for $X \in \mathfrak{sl}_2$

$$\frac{d}{dt}\Big|_0 \langle e^{tX} f, e^{tX}\varphi\rangle = 0$$

so

$$\langle Xf, \varphi\rangle = -\langle f, X\varphi\rangle.$$

That is, for a representation of $\mathfrak{sl}_2$ compatible with one from $\mathrm{SL}_2(\mathbb{R})$, the operators coming from the former should be *skew-symmetric*. The operators $x = -\Delta^{\mathbb{R}^n}/2$ and $y = r^2/2$ are merely symmetric. Renormalize by $x = i\Delta^{\mathbb{R}^n}/2$ and $y = ir^2/2$, to induce skew symmetry, while retaining the relation $[x, y] = h$. Next, we pick new coordinates for $\mathfrak{sl}_2$, better suited to a representation coming from $\mathrm{SL}_2(\mathbb{R})$.

The basis elements of $\mathfrak{sl}_2$

$$x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

correspond to a decomposition of $\mathrm{SL}_2(\mathbb{R})$ as

$$\mathrm{SL}_2(\mathbb{R}) = (\text{upper triangular}) \cdot (\text{diagonal}) \cdot (\text{lower triangular}).$$

Notably each one of these components is *not* compact.

Supposing that the oscillator representation *comes* from a representation of the underlying group $\mathrm{SL}_2(\mathbb{R})$, we are more likely to find an operator with rapidly decaying eigenfunctions coming from a *compact* group.

We look to the *Iwasawa decomposition* of $\mathrm{SL}_2(\mathbb{R})$, letting P denote the subgroup of upper triangular matrices and $K = \mathrm{SO}(2)$ the maximal compact subgroup,

$$\mathrm{SL}_2(\mathbb{R}) = PK.$$

The Lie algebra $\mathfrak{k}$ of the maximal compact subgroup K is spanned by

$$\theta = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = x - y.$$

Its image under the oscillator representation is $\frac{i}{2}(r^2 - \Delta^{\mathbb{R}^n})$.

3.4.2 Spectral theory of $L^2(\mathbb{R}^n)$ via θ

For this section, we replace θ by $-2i\theta$ so that its image $\theta = r^2 - \Delta^{\mathbb{R}^n}$ is a *positive* operator, thus amenable to Friedrichs' theorem. When we determine the eigenvectors, we will renormalize so as to fit with the weight theory developed in the last chapter.

We take the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ of smooth, rapidly decaying functions as the domain of the unbounded operator $\theta = r^2 - \Delta^{\mathbb{R}^n}$. For now, we just think of $\mathscr{S}(\mathbb{R}^n)$ as a complex vectorspace. Later we will topologize it with a countable family of seminorms. Letting $C_c^\infty(\mathbb{R}^n)$ denote the space of smooth, compactly supported test functions, we define

$$L^2(\mathbb{R}^n) = \text{completion of } C_c^\infty(\mathbb{R}^n) \text{ with respect to } |f|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |f|^2 \, d\mu.$$

On $\mathscr{S}(\mathbb{R}^n)$, multiplication by r^2 and application of $\Delta^{\mathbb{R}^n}$ are both *symmetric*, so that θ is too,

$$\langle \theta f, g\rangle = \langle f, \theta g\rangle \quad \text{for all } f, g \in \mathscr{S}(\mathbb{R}^n).$$

Further, multiplication by r^2 is a *positive* operator, and integration by parts shows $\Delta^{\mathbb{R}^n}$ is *negative*, so θ is *positive*

$$\langle \theta f, f\rangle \geq 0 \quad \text{for all } f \in \mathscr{S}(\mathbb{R}^n).$$

Thus, by Friedrichs' theorem, θ has a *positive, self-adjoint* extension $\widetilde{\theta}$ defined on a dense subspace of the Sobolev-like space

$$\mathfrak{B}^1(\mathbb{R}^n) = \text{completion of } \mathscr{S}(\mathbb{R}^n) \text{ with respect to } |f|^2_{\mathfrak{B}^1} = \langle \theta f, f\rangle.$$

Further, the extension $\widetilde{\theta}$ is characterized by the relation with its resolvent,

$$\langle f, g\rangle = \langle \theta f, \widetilde{\theta}^{-1} g\rangle \quad (\text{for } f \in \mathscr{S}(\mathbb{R}^n) \text{ and } g \in L^2(\mathbb{R}^n)).$$

The resolvent $\widetilde{\theta}^{-1}$ is a positive, self-adjoint, continuous map $L^2(\mathbb{R}^n) \to \mathfrak{B}^1(\mathbb{R}^n)$, surjecting onto the domain of $\widetilde{\theta}$.

Compactness of the inclusion $\mathfrak{B}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$

To show that $L^2(\mathbb{R}^n)$ has an orthonormal basis of eigenvectors for the extension $\widetilde{\theta}$, it suffices to prove that the resolvent $\widetilde{\theta}^{-1}$ is compact. To prove that $\widetilde{\theta}^{-1}$ is compact, it suffices to prove that the inclusion $\mathfrak{B}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is compact.

Unlike our proof of compactness of the relevant inclusion on the circle and the sphere, we must now account for the non-compactness of $L^2(\mathbb{R}^n)$. Non-compactness of $\mathbb{R}^n$ imposes a constraint on the decay of a function, as is manifest in the proof.

Claim 7. The inclusion $\mathfrak{B}^1(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is compact.

Proof. Compactness of the inclusion $\mathfrak{B}^1(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is equivalent to the image of the unit $\mathfrak{B}^1(\mathbb{R}^n)$-ball being totally bounded in $L^2(\mathbb{R}^n)$. We reduce the proof of total boundedness to an invocation of Rellich compactness on $\mathbb{T}^n$, paired with an estimate on the smoothly truncated tail.

Take $f \in \mathscr{S}(\mathbb{R}^n)$ with $|f|^2_{\mathfrak{B}^1} = \langle \theta f, f\rangle \leq 1$, fix $\varepsilon > 0$ and N such that $N^2 > 1/\varepsilon$. Define a smooth cut-off function $\varphi_N : \mathbb{R}^n \to [0, 1]$ identically 0 on the closed radius N ball B_N

about the origin in $\mathbb{R}^n$, identically 1 on the complement of the radius $N+1$ ball B_{N+1}, and smoothly varying between 0 and 1 for $x \in B_{N+1} - B_N$. Now write $f = f_1 + f_2$, with $f_1 = \varphi_N f$ supported on B_{N+1} and $f_2 = (1-\varphi_N) f$ supported on $\mathbb{R}^n - B_N$.

The ball B_{N+1} sits in the box $T = [-(N+1), N+1]^n$. Identify T with the n-torus $\mathbb{T}^n$ by identifying opposite edges, and identify f_1 with a function on $\mathbb{T}^n$. Now note that

$$\begin{aligned} |f_1|^2_{\mathfrak{B}^1} &= \langle r^2 f_1, f_1 \rangle - \langle \Delta^{\mathbb{R}^n} f_1, f_1 \rangle \\ &= (N+1)^n |f_1|^2_{L^2(\mathbb{T}^n)} - \langle \Delta^{\mathbb{R}^n} f_1, f_1 \rangle_{L^2(\mathbb{T}^n)} \\ &\leq (N+1)^n |f_1|^2_{H^1(\mathbb{T}^n)}. \end{aligned}$$

Thus, the image of the smoothly truncated unit $\mathfrak{B}^1$ ball is contained in the radius $(N+1)^n$ ball in $H^1(\mathbb{T}^n)$. By Rellich's lemma on $\mathbb{T}^n$, the radius $(2N+2)^n$ ball in $H^1(\mathbb{T}^n)$ is covered by finitely many radius ε balls in $L^2(\mathbb{T}^n)$. Now identify $\mathbb{T}^n$ with $T \subset \mathbb{R}^n$ and identify $L^2(\mathbb{T}^n)$ isometrically with the subspace in $L^2(\mathbb{R}^n)$ of functions supported on T. Then the smoothly truncated unit $\mathfrak{B}^1$ ball is covered by finitely many ε balls in $L^2(\mathbb{R}^n)$.

Now we show that the totality of smoothly cut-off tails

$$\{f_2 \in \mathscr{S}(\mathbb{R}^n) : f_2 = (1-\varphi_N) f \text{ for } |f|_{\mathfrak{B}^1} < 1\}$$

is contained in a single ε-ball in $L^2(\mathbb{R}^n)$. Compute directly for such an f,

$$\begin{aligned} |f_2|^2_{L^2(\mathbb{R}^n)} &= \int_{|x|\geq N} |f_2(x)|^2 \, \mathrm{d}\mu(x) \\ &\leq \frac{1}{N^2} \int_{|x|\geq N} |r \cdot f_2(x)|^2 \, \mathrm{d}\mu(x) && \text{where } r = |x| \\ &\leq \frac{1}{N^2} \int_{\mathbb{R}^n} r^2 f(x) \cdot \overline{f}(x) \, \mathrm{d}\mu(x) \\ &\leq \frac{1}{N^2} \int_{\mathbb{R}^n} (r^2 - \Delta^{\mathbb{R}^n}) f(x) \cdot \overline{f}(x) \, \mathrm{d}\mu(x) && \text{since } \Delta^{\mathbb{R}^n} \text{ is } \textit{negative} \\ &\leq \frac{1}{N^2} |f|^2_{\mathfrak{B}^1} \leq \frac{1}{N^{2n}} < \varepsilon && \text{since } |f|^2_{\mathfrak{B}^1} < 1. \end{aligned}$$

Thus the tails are all contained in the ε ball in $L^2(\mathbb{R}^n)$, completing the proof of total boundedness. Thus $\mathfrak{B}^1(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is compact. □

Consequently the resolvent $\widetilde{\theta}^{-1} : L^2(\mathbb{R}^n) \to \mathfrak{B}^1(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is compact, being the composition of a continuous map and a compact inclusion. By the spectral theorem for compact, self adjoint operators $L^2(\mathbb{R}^n)$ has an orthonormal basis of eigenfunctions of $\widetilde{\theta}^{-1}$, in turn eigenfunctions of $\widetilde{\theta}$.

Sobolev-like regularity

To prove that the eigenfunctions of $\widetilde{\theta}$ coincide with those of θ, it suffices to prove that such functions are in $\mathscr{S}(\mathbb{R}^n)$, since then evaluation of $\widetilde{\theta}$ on $\mathscr{S}(\mathbb{R}^n)$ is tantamount to evaluating θ.

Let $F \in \mathfrak{B}^1(\mathbb{R}^n)$ be an eigenfunction of $\widetilde{\theta}$ with eigenvalue $\lambda > 0$. Then

$$\widetilde{\theta} F = \lambda F,$$

so that

$$F = \widetilde{\theta}^{-1} \widetilde{\theta} F \in \widetilde{\theta}^{-1} \mathfrak{B}^1(\mathbb{R}^n).$$

Define the $+k$ Sobolev-like space,

$$\mathfrak{B}^k = \text{completion of } \mathscr{S}(\mathbb{R}^n) \text{ with respect to } |f|^2_{\mathfrak{B}^k} = \langle \theta^k f, f \rangle.$$

By induction, we have just shown that the eigenfunction F lies in $\mathfrak{B}^k(\mathbb{R}^n)$ for every $k \geq 0$. That is, $F \in \mathfrak{B}^\infty(\mathbb{R}^n) = \bigcap \mathfrak{B}^k(\mathbb{R}^n) = \lim \mathfrak{B}^k(\mathbb{R}^n)$. Thus, to prove that the eigenfunction F is in $\mathscr{S}(\mathbb{R}^n)$, it suffices to prove that $\mathfrak{B}^\infty = \mathscr{S}(\mathbb{R}^n)$.

Recall that $\theta = x - y = r^2 - \Delta^{\mathbb{R}^n}$. Since r^2 and $\Delta^{\mathbb{R}^n}$ do not commute, the definition of the $\mathfrak{B}^k(\mathbb{R}^n)$ norm is difficult to work with. We can get away with topologizing $\mathfrak{B}^k(\mathbb{R}^n)$ with a simpler family of seminorms, as follows. Define the *creation* and *annihilation* operators

$$C_i = x_i - \frac{\partial}{\partial x_i} \quad \text{and} \quad A_i = x_i + \frac{\partial}{\partial x_i} \quad \text{for } 1 \leq i \leq n.$$

By Leibniz's rule, $\frac{\partial}{\partial x_i} x_j = \delta_{ij} + x_j \frac{\partial}{\partial x_i}$, we have the commutation relations

$$[A_i, A_j] = [C_i, C_j] = 0 \quad \text{and} \quad [C_i, A_j] = -2\delta_{ij}.$$

Furthermore, we have the following expression for θ.

$$\theta = \sum_{i=1}^n (C_i A_i + 1) = \sum_{i=1}^n (A_i C_i - 1) = \frac{1}{2} \sum_{i=1}^n C_i A_i + A_i C_i.$$

For convenience, set

$$C = \{C_1, ..., C_n\} \quad \text{and} \quad A = \{A_1, ..., A_n\}.$$

Working in the noncommutative polynomial algebra $W = \mathbb{C}[C, A]$ generated by the C_i and A_i, give each C_i and A_i *degree* 1. Define the degree of a monomial in $\mathbb{C}[C, A]$ as the *total degree*, the sum of the degree of the constituent productands. Then the degree of a polynomial is that of its highest degree monomial. Let $W^{\leq m}$ be the (vector) subspace of polynomials in W with degree at most m.

For any monomial $w \in W$, define a seminorm on $C_c^\infty(\mathbb{R}^n)$

$$\mu_w(f) = |\langle wf, f \rangle|.$$

The following claim demonstrates that $\mathfrak{B}^m(\mathbb{R}^n)$ embeds continuously into the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to the family of seminorms $M_{2m} = \{\mu_w : w \text{ a monomial in } W^{\leq 2m}\}$.

Claim 8. For $f \in C_c^\infty(\mathbb{R}^n)$ and $w \in M_{2m}$, we have

$$\mu_w(f)^2 \leq c_m |f|^2_{\mathfrak{B}^m} \quad (\text{with } c_m \text{ independent of } f).$$

Proof. Monomials of degree 0 are constants, and the claim is obvious. When $m = 1$, the the monomials in $W^{\leq 2}$ are of the form $C_iA_j = 2\delta_{ij} + A_jC_i$ and $C_iC_j = C_jC_i = (A_iA_j)^*$.. Thus, it suffices to check the claim for C_iA_j and C_iC_j. Let B_j be either C_j or A_j. Compute first

$$\begin{aligned}|\langle C_iB_jf, f\rangle| = |\langle(B_jf, A_if\rangle| &\leq |\langle B_jf, B_jf\rangle|^{1/2}|\langle A_if, A_if\rangle|^{1/2}\\ &= |\langle B_j^*B_jf, f\rangle|^{1/2}|\langle C_iA_if, f\rangle|^{1/2}.\end{aligned}$$

Thus, it suffices to check the case that $i = j$. Compute

$$\begin{aligned}|\langle C_jA_jf, f\rangle| &= |\langle(C_jA_j + 1)f, f\rangle - \langle f, f\rangle|\\ &\leq \langle(C_jA_j + 1)f, f\rangle\\ &\leq \sum_{j=1}^{n}\langle(C_jA_j + 1)f, f\rangle = \langle\theta f, f\rangle = |f|^2_{\mathfrak{B}^1},\end{aligned}$$

as claimed.

Using the commutation relations, any monomial w in $C_1, ..., C_n$ and $A_1, ..., A_n$ of degree $2n$ may be reduced to a monomial of the form $C^\alpha A^\beta$ + (lower order terms) where α and β are multi-indices such that $|\alpha| + |\beta| = 2n$. Supposing the result for $n - 1$, we have

$$|\langle wf, f\rangle| \leq |\langle C^\alpha A^\beta, f\rangle| + c_{n-1}|f|^2_{\mathfrak{B}^{n-1}}.$$

Supposing there is some i which $\alpha_i, \beta_i \geq 1$ compute (letting e_i be the ith standard basis vector)

$$\begin{aligned}|\langle C^\alpha A^\beta f, f\rangle| &= |\langle C_iC^{\alpha-e_i}A^{\beta-e_i}A_if, f\rangle\\ &= |\langle C^{\alpha-e_i}A^{\beta-e_i}A_if, A_if\rangle|\\ &\leq c_{n-1}|A_if|^2_{\mathfrak{B}^{n-1}(\mathbb{R}^n)}\\ &= c_{n-1}\langle\theta^{n-1}A_if, A_if\rangle\\ &= c_{n-1}\langle C_i\theta^{n-1}A_if, f\rangle.\end{aligned}$$

Then, since $C_i\theta^{n-1}A_i = C_iA_i\theta^{n-1}$ + (lower order terms), and since $\sum_{i=1}^n(C_iA_i\theta^{n-1} + \theta^{n-1}) = \theta^n$, we have

$$\begin{aligned}c_{n-1}\langle C_i\theta^{n-1}A_if, f\rangle &\leq c_{n-1}\langle\theta^n f, f\rangle + c_{n-1}^2|f|^2_{\mathfrak{B}^{n-1}}\\ &= c_{n-1}|f|^2_{\mathfrak{B}^n} + c_{n-1}^2|f|^2_{\mathfrak{B}^{n-1}}\\ &\leq c_n|f|^2_{\mathfrak{B}^n}.\end{aligned}$$

In the case that there is no common nonzero index among α and β then C^α and A^β commute. Still assuming both $|\alpha|$ and $|\beta|$ are nonzero, compute

$$\begin{aligned}|\langle C^\alpha A^\beta f, f\rangle| &= |\langle A^\beta f, A^\alpha f\rangle|\\ &\leq \langle A^\beta f, A^\beta f\rangle^{1/2}\langle A^\alpha f, A^\alpha f\rangle^{1/2}\\ &= \langle C^\beta A^\beta f, f\rangle^{1/2}\langle C^\alpha A^\alpha f, f\rangle^{1/2}.\end{aligned}$$

bringing us back to the case where there are nonzero common indices. Last, suppose $|\alpha| = 0$ so that $|\beta| = 2n$. By pigeonhole, there is some $\beta_i \geq 2$. Then

$$\begin{aligned}|\langle A^\beta f, f\rangle| &= |\langle A_i A^{\beta-2e_i} A_i f, f\rangle| \\ &\leq \langle A^{\beta-2e_i} A_i f, A^{\beta-2e_i} A_i f\rangle^{1/2} \langle C_i f, C_i f\rangle^{1/2} \\ &= \langle C^{\beta-2e_i} A^{\beta-2e_i} A_i f, f\rangle^{1/2} \langle A_i C_i f, f\rangle^{1/2}\end{aligned}$$

to which we can apply the last computation. This covers all possible cases, proving the claim. □

To compare the topology generated by the μ_w and that on $\mathscr{S}(\mathbb{R}^n)$, we introduce the family of *Schwartz seminorms*. For a multi-index $\beta = (\beta_1, ..., \beta_\ell)$, let

$$\partial_\beta = \frac{\partial^{|\beta|}}{\partial x_{\beta_1} ....\partial x_{\beta_\ell}}.$$

Since any $f \in \mathscr{S}(\mathbb{R}^n)$ and all of its derivatives decay faster than every polynomial, $(1 + |x|^m)\partial_\beta f$ is bounded for every integer m and multi-index β. Now, for all nonnegative integers ℓ, m, define a norm on $C_c^\infty(\mathbb{R}^n)$

$$v_{\ell,m}(f)^2 = \sum_{|\beta|\leq\ell} \sup_{x\in\mathbb{R}^n} |(1 + |x|^2)^m \partial_\beta f(x)|^2.$$

Define a composite norm on $\mathscr{S}(\mathbb{R}^n)$

$$|f|^2_{\mathscr{S}^k} = \sum_{\ell+m\leq k} v_{\ell,m}(f)^2,$$

and set

$$\mathscr{S}^k(\mathbb{R}^n) = \mathscr{S}(\mathbb{R}^n) \text{ topologized by } |f|^2_{\mathscr{S}^k}.$$

We topologize Schwartz space as the limit

$$\mathscr{S}(\mathbb{R}^n) = \lim_k \mathscr{S}^k(\mathbb{R}^n)$$

Now, to show that $\mathfrak{B}^\infty(\mathbb{R}^n) = \mathscr{S}(\mathbb{R}^n)$ it suffices to show that for each k, there is some s such that

$$\mathfrak{B}^s(\mathbb{R}^n) \subset \mathscr{S}^k(\mathbb{R}^n).$$

But, by the last claim, we know that the $\mathfrak{B}^s(\mathbb{R}^n)$ norm dominates the family of seminorms $\mu_{w_{2s}}$ generated by monomials of degree $2s$ in C_i and A_i. Thus, to prove $\mathfrak{B}^s(\mathbb{R}^n) \subset \mathscr{S}^k(\mathbb{R}^n)$, it suffices to prove that the seminorms $\mu_{w_{2s}}$ dominate the Schwartz norm on $\mathscr{S}^k(\mathbb{R}^n)$.

Claim 9. For any k there is some s, and some finite collection of monomials $w_{2s+2k,i}$ each with degree at most $2s + 2k$ so that

$$|f|_{\mathscr{S}^k(\mathbb{R}^n)} \leq \sum_i \mu_{w_{2s+2k,i}}(f) \quad \text{for all } f \in C_c^\infty(\mathbb{R}^n).$$

Proof. To prove the case $k = 0$, we use the classical Sobolev embedding theorem: for integral $s > n/2$, and for any $f \in C_c^\infty(\mathbb{R}^n)$, we have

$$|f|^2_{C^0_c} = \sup_{x\in\mathbb{R}^n} |f(x)| \le c_n \sum_{|\alpha|\le s} |\partial^\alpha f|^2_{L^2(\mathbb{R}^n)} = c_n |f|^2_{H^s(\mathbb{R}^n)}.$$

First, note that $|f|_{C^0_c} = v_{0,0}(f)$ and that $\frac{\partial}{\partial x_i} = 1/2(A_i - C_i)$. Thus, from Sobolev, we have

$$\begin{aligned}
|f|^2_{C^0_c} &\le c_n \sum_{|\alpha|\le s} |\partial^\alpha f|^2_{L^2(\mathbb{R}^n)} \\
&= c_n \sum_{|\alpha|\le s} 1/2^{2|\alpha|} \langle (C-A)^\alpha f, (C-A)^\alpha f\rangle \\
&= c_n \sum_{|\alpha|\le s} 1/2^{2|\alpha|} \langle -(C-A)^{2\alpha} f, f\rangle \\
&= \sum_i \mu_{w_{2s,i}}(f),
\end{aligned}$$

since of the $(C^\alpha - A^\alpha)^2$ expands to a sum of monomials of degree at most $2|\alpha| \le 2s$.

Now take $s > k + n/2$. We want to dominate the norm

$$\begin{aligned}
|f|^2_{\mathscr{S}^k} &= \sum_{|\alpha|+m\le k} \sup_{x\in\mathbb{R}^n} |(1+|x|^2)^m \partial^\alpha f|^2 \\
&= |f|^2_{\mathscr{S}^{k-1}} + \sum_{|\alpha|\le k} v_{0,0}((1+|x|^2)^{k-|\alpha|}\partial^\alpha f) \\
&\le \sum_i \mu_{w_{2s+2k-2,i}}(f) + \sum_{|\alpha|\le k} v_{0,0}((1+|x|^2)^{k-|\alpha|}\partial^\alpha f)
\end{aligned}$$

Note that $|x|^2 = \sum_{i=1}^n x_i^2 = 1/2^n \sum_{i=1}(C_i + A_i)^2$. Let v_j be the degree $2k$ polynomial $(1 + (C_j + A_j)^2)^{|\alpha|-k}(C-A)^\alpha$. Then we have, in each of the latter summands in the last display, using the base case

$$\begin{aligned}
v_{0,0}((1+|x|^2)^{|\alpha|-k}\partial^\alpha f) &\le c_n \sum_i \mu_{w_{2s,i}}((1+|x|^2)^{|\alpha|-k}\partial^\alpha f) \\
&\le c_n 1/2^k \sum_i \sum_{j=1}^n \mu_{w_{2s,i}}(v_j f) \\
&= c_n' \sum_i \sum_{j=1}^n \langle v_j^* w_{2s,i} v_j f, f\rangle = \sum_i \mu_{w'_{2s+2k,i}}(f).
\end{aligned}$$

The monomials $w'_{2s+2k,i}$ are obtained by expanding the polynomials $v_j^* w_i v_j$. Each has degree at most $2s + 2k$ since v_j has degree k and $w_{2s,i}$ has degree $2s$. Applying this to each of the summands in the penultimate display, we find that

$$|f|^2_{\mathscr{S}^k(\mathbb{R}^n)} \le \sum_i \mu_{w_{2s+2k-2,i}}(f) + \sum_i \mu_{w'_{2k+2k,i}}(f) = \sum_i \mu_{w_{2s+2k}}(f),$$

as claimed. □

As a result, $\mathfrak{B}^s(\mathbb{R}^n)$ embeds continuously in $\mathscr{S}^k(\mathbb{R}^n)$ for $s > k + n/2$. Thus, $\mathfrak{B}^\infty(\mathbb{R}^n)$ embeds continuously into $\mathscr{S}(\mathbb{R}^n)$ showing that $\mathfrak{B}^\infty(\mathbb{R}^n) = \mathscr{S}(\mathbb{R}^n)$.

We have shown that any eigenfunction of $\widetilde{\theta}$ is in $\mathfrak{B}^\infty(\mathbb{R}^n)$, and thus such an eigenfunction is in $\mathscr{S}(\mathbb{R}^n)$. Since $\widetilde{\theta}$ and θ agree on their common domain $\mathscr{S}(\mathbb{R}^n)$, we have shown that any eigenfunction of $\widetilde{\theta}$ is an eigenfunction of θ.

Consequently, there is an orthonormal basis of $L^2(\mathbb{R}^n)$ consisting of eigenfunctions of θ.

3.4.3 Complete determination of eigenvectors

Reintroduction of the oscillator representation

To determine the eigenfunctions of θ, we will use the lowest weight theory developed in the last section. To refresh one's memory, recall that on $\mathbb{C}[x_1, \ldots, x_n]$

- The operators $x = r^2/2$, $y = -\Delta^{\mathbb{R}^n}/2$ and $h = E + n/2$ satisfy the commutation relations
$$[x, y] = h \quad [h, x] = 2x \quad [h, y] = -2y.$$
and thus generate an $\mathfrak{sl}_2$ representation.
- Given a weightvector v for h with weight λ, the vector xv is again a weightvector, now with weight $\lambda + 2$, and yv is a weightvector with weight $\lambda - 2$.
- The representation generated by a lowest weightvector of weight λ (that is, an λ-eigenfunction of h annihilated by y) is irreducible, and further is uniquely determined by λ up to $\mathfrak{sl}_2$ isomorphism.
- Any representation generated by a λ-weightvector that is annihilated by a power m of y is a finite direct sum of lowest weight representations with weights among $\{\lambda, \lambda - 2, ..., \lambda - 2m\}$ occurring at most once. Each such lowest weight representation is irreducible and stable under the action of $\mathfrak{sl}_2$.

Return to the normalization
$$x \longmapsto ir^2/2$$
$$y \longmapsto i\Delta^{\mathbb{R}^n}/2$$

so that $\theta = i/2(r^2 - \Delta^{\mathbb{R}^n})$

To determine all of the weightvectors of θ, we need raising and lowering operators R, L satisfying the $\mathfrak{sl}_2$-commutation relations
$$[R, L] = \theta \quad [\theta, R] = 2R \quad [\theta, L] = -2L.$$

That is, we need are looking for eigenvectors of the operator

$$[\theta, \cdot] : X \longmapsto \theta X - X\theta \quad \text{on } \mathfrak{sl}_2$$

To find eigenvectors, we diagonalize θ, now working in the complexified Lie algebra $\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{sl}_2 \otimes \mathbb{C}$,

$$\theta = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \cdot \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \cdot \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}^{-1} = c \cdot ih \cdot c^{-1}.$$

The $\mathfrak{sl}_2$ relations show that the eigenspaces of ih under $[ih, \cdot]$ are

$$0\text{-eigenspace} = \mathbb{C}h \quad 2i\text{-eigenspace} = \mathbb{C}x \quad -2i\text{-eigenspace} = \mathbb{C}y.$$

Thus, changing coordinates, the corresponding eigenspaces for $[\theta, \cdot]$ are

$$0\text{-eigenspace} = \mathbb{C}\theta \quad 2\text{-eigenspace} = \mathbb{C}cxc^{-1} \quad -2\text{-eigenspace} = \mathbb{C}cyc^{-1}.$$

Define the *raising* and *lowering* operators

$$R = cxc^{-1} = \frac{1}{2}\begin{bmatrix} -i & 1 \\ 1 & i \end{bmatrix} \qquad L = cyc^{-1} = \frac{1}{2}\begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}.$$

These satisfy the relations

$$[\theta, R] = 2iR \quad [\theta, L] = -2iL \quad [R, L] = -i\theta.$$

Under the oscillator representation,

$$\begin{aligned} R = -\frac{1}{2}(ih + 2x + 2y) &\longmapsto \frac{i}{2}(-n/2 - E + r^2 + \Delta^{\mathbb{R}^n}) \\ L = \frac{1}{2}(ih + 2x + 2y) &\longmapsto \frac{i}{2}(n/2 + E + r^2 + \Delta^{\mathbb{R}^n}) \\ \theta = x - y &\longmapsto \frac{i}{2}(r^2 - \Delta^{\mathbb{R}^n}). \end{aligned}$$

Analysis of weightspaces

As computed, $r^2 - \Delta^{\mathbb{R}^n}$ is a *positive*, symmetric operator. Consequently, its eigenvalues are real and strictly positive. Thus, the eigenvalues of θ are of the form $i\lambda$ for $\lambda > 0$. Given any weightvector f with weight $i\lambda$, the vector $L^j f$ has weight $i(\lambda - 2j)$. For large enough j, $\lambda - 2j \leq 0$, so that some power of L annihilates f.

From the last section, the representation generated by a λ-weightvector annihilated by a power of L is a finite direct sum of *lowest weightspaces*, each with (lowest) weight among $\{\lambda, \lambda - 2, ...\}$ occurring at most once. Consequently to determine all weightvectors (that is, eigenfunctions of θ), it suffices to determine all of the *lowest weightvectors*.

We must solve the simultaneous equations

$$\begin{aligned} \theta f &= i\lambda f \quad \text{for } \lambda > 0 \\ Lf &= 0 \end{aligned}$$

for $f \in \mathscr{S}(\mathbb{R}^n)$. A candidate solution to the first is the *Gaussian*

$$g_a(x) = e^{-a|x|^2/2}.$$

We check

$$\Delta^{\mathbb{R}^n} g_a(x) = (a^2r^2 - na)g_a(x)$$

Thus,

$$i/2(r^2 - \Delta^{\mathbb{R}^n})g_a(x) = i/2(r^2 + an - a^2r^2)g_a(x),$$

so we need $a^2 = 1$ (to cancel the r^2 term) and $an > 0$ (for square integrability). The only possible solution of this form is

$$g(x) = g_1(x) = e^{-|x|^2/2} \quad \text{with eigenvalue } in/2.$$

Knowing that the Gaussian g is an weightvector of θ, we look for solutions of the form Pg for some function P of moderate growth. From the general identity

$$\Delta^{\mathbb{R}^n}(hf) = \Delta^{\mathbb{R}^n} h \cdot f + 2\sum_{i=1}^{n} h_i f_i + h \cdot \Delta^{\mathbb{R}^n} f \quad (\text{for } h, f \in \mathscr{S}(\mathbb{R}^n))$$

compute (recalling that $E = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i}$),

$$\begin{aligned}(r^2 - \Delta^{\mathbb{R}^n})(Pg) &= r^2Pg - (\Delta^{\mathbb{R}^n} P \cdot g - 2(\sum_{i=1}^{n} x_iP_i)g + P(-n + r^2)g) \\ &= r^2Pg - \Delta^{\mathbb{R}^n} P \cdot g + 2EP \cdot g + nPg - r^2Pg \\ &= -\Delta^{\mathbb{R}^n} P \cdot g + (2E + n)P \cdot g.\end{aligned}$$

Thus, if P is *annihilated* by $\Delta^{\mathbb{R}^n}$ and a λ-eigenfunction of E then Pg is a $i(\lambda + n/2)$-weightvector of θ. That is, P must be *harmonic* and *homogeneous*. In the section on spherical harmonics, we showed that *homogeneous* functions are of moderate growth, and that *harmonic* functions of moderate growth are actually *polynomials*. Thus, we have just discovered a large class of weightvectors

$$Pg = i(d + n/2) \text{ weightvector of } \theta \quad \text{for any } P \in \mathfrak{H}^d.$$

Next, we show that Pg is actually a *lowest* weightvector. This amounts to computing

$$L(Pg) = i/2(n + 2E + r^2 + \Delta^{\mathbb{R}^n})(Pg).$$

This computation can be simplified, since we have already computed

$$\theta(Pg) = i/2(r^2 - \Delta^{\mathbb{R}^n})(Pg) = i(n/2 + d)Pg,$$

so

$$\Delta^{\mathbb{R}^n}(Pg) = (r^2 - n - 2d)Pg.$$

Now (using $Eg = -r^2 g$ in the fourth line)

$$\begin{aligned} L(Pg) &= i/2(n + 2E + 2r^2 - n - 2d)Pg \\ &= i/2(-2dPg + 2(EP \cdot g + P \cdot Eg) + 2r^2 Pg) \\ &= i/2(-2dPg + 2dPg - 2r^2 Pg + 2r^2 Pg) = 0, \end{aligned}$$

as claimed.

Thus, the *harmonic homogeneous* degree d polynomial-weighted Gaussian Pg is a *lowest* weightvector of θ, with weight $i(n/2 + d)$. The $i(n/2+d)$-lowest weight representation generated by Pg is *irreducible*, and is uniquely determined by its weight, up to $\mathfrak{sl}_2$ isomorphism. With $M_{n/2+d}$ being any representative of that isomorphism class, we have (recalling that R is the *raising* operator)

$$M_{n/2+d} \approx \mathfrak{sl}_2 \cdot Pg = \bigoplus_{a \geq 0} \mathbb{C} R^a Pg \qquad \text{(as } \mathfrak{sl}_2 \text{ repn's)}.$$

The Gaussian $g(x) = e^{-|x|^2/2}$ is $\mathrm{SO}(n)$ invariant, since $\mathrm{SO}(n)$ acts on $\mathbb{R}^n$ by isometries. Thus, $\mathrm{SO}(n)$ stabilizes the weightspaces $\mathfrak{H}^d \otimes g = \{Pg : P \in \mathfrak{H}^d\}$, and acts transitively thereupon. Thus, in $\mathscr{S}(\mathbb{R}^n)$, we have the direct sum of irreducibles

$$\bigoplus_{d \geq 0} \mathfrak{H}^d \otimes M_{n/2+d} \subset \mathscr{S}(\mathbb{R}^n) \qquad \text{(as } \mathrm{SO}(n) \times \mathfrak{sl}_2 \text{ repn's)}.$$

Next, we show that this subrepresentation exhausts the list of θ-eigenfunctions.

Complete determination of eigenvectors

We know that any λ-eigenfunction lies in a finite direct sum of lowest weightspaces. Thus, to determine all θ eigenfunctions, it suffices to determine those which are annihilated by L.

Suppose $f \in \mathscr{S}(\mathbb{R}^n)$ is a $i\lambda$-lowest weightvector, so that

$$\theta f = \lambda f$$
$$Lf = 0.$$

We show that f is of the form Pg, with P a harmonic homogeneous polynomial and g the Gaussian, by showing that $f \cdot e^{|x|^2/2}$ is harmonic and homogeneous. From the section on spherical harmonics, we know that harmonic homogeneous functions are actually *polynomials*.

Set $g_{-1}(x) = e^{|x|^2/2}$ and compute

$$\begin{aligned} \Delta^{\mathbb{R}^n}(g_{-1} f) &= \Delta^{\mathbb{R}^n} g_{-1} \cdot f + 2\sum_{i=1}^{n} x_i g \cdot f_i + g_{-1} \Delta^{\mathbb{R}^n} f \\ &= (n + r^2) g_{-1} f + 2Ef \cdot g_{-1} + g_{-1} \Delta^{\mathbb{R}^n} f \\ &= g_{-1}(n + 2E + r^2 + \Delta^{\mathbb{R}^n}) f = g_{-1} \cdot \frac{4}{i} Lf = 0. \end{aligned}$$

Thus, $g_{-1}f$ is *harmonic.*

For homogeneity, recall

$$i/4Lf = (n + 2E + r^2 + \Delta^{\mathbb{R}^n})f = 0$$

so $Ef = (-n/2 - r^2/2 - \Delta^{\mathbb{R}^n}/2)f$. Then, compute

$$\begin{aligned} E(g_{-1}f) &= Eg_{-1} \cdot f + g_{-1}Ef \\ &= r^2 g_{-1}f + ig_{-1}(-n/2 + r^2/2 - \Delta^{\mathbb{R}^n}/2)f \\ &= g_{-1}(r^2/2 - n/2 - \Delta^{\mathbb{R}^n}/2)f \\ &= g_{-1}(-n/2 - i\theta)f = (-n/2 + \lambda)g_{-1}f. \end{aligned}$$

Thus, $g_{-1}f$ is *homogeneous*, of degree $\lambda - n/2$.

To apply the Fourier transform to $g_{-1}f$ as a *distribution* it suffices to check that the homogeneity degree $\lambda - n/2$ is nonnegative, since then $g^{-1}f$ is of *moderate growth* and thus defines a tempered distribution. That is, we must prove that the smallest eigenvalue is at least $n/2$.

Recall that we defined the pairwise commuting family of *annihilation* operators

$$A_i = x_i + \frac{\partial}{\partial x_i} \qquad \text{for } 1 \le i \le n.$$

We saw that

$$\theta = i/2 \sum_{i=1}^{n} (A_i^* A_i + 1) \qquad \text{with } A_i^* = C_i = x_i - \frac{\partial}{\partial x_i}.$$

For the $i\lambda$-eigenfunction f of $\theta = i/2(r^2 - \Delta^{\mathbb{R}^n})$, compute

$$\begin{aligned} \lambda |f|^2_{L^2(\mathbb{R}^n)} &= \langle \frac{1}{2}(r^2 - \Delta^{\mathbb{R}^n})f, f\rangle = 1/2(\sum_{i=1}^{n} \langle A_i^* A_i f, f\rangle + n|f|^2_{L^2(\mathbb{R}^n)}) \\ &= 1/2(\sum_{i=1}^{n} |A_i f|^2_{L^2(\mathbb{R}^n)} + n|f|^2_{L^2(\mathbb{R}^n)}) \ge n/2|f|^2_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Clearing a factor of $|f|^2_{L^2(\mathbb{R}^n)}$, we find that $\lambda \ge n/2$, as claimed.

As in the section on spherical harmonics, positive homogeneous functions of degree ≥ 0 are of *moderate* growth, and thus define *tempered* distributions. The Fourier transform of a *harmonic* tempered distribution is supported at 0, and is thus a finite sum of derivatives of Dirac delta distributions. Fourier inversion shows that such distributions are *polynomials.* Thus, $g_{-1}f$ is a *harmonic, homogeneous polynomial.*

Consequently, the lowest weightvector f is itself of the form Pg for P a harmonic homogeneous polynomial and $g(x) = e^{-|x|/2}$.

Conclusion

Every eigenfunction of θ in $\mathscr{S}(\mathbb{R}^n)$ is annihilated by some power of L. Consequently, every eigenfunction in $\mathscr{S}(\mathbb{R}^n)$ is a finite linear combination of $\mathfrak{sl}_2$ translates of eigenfunctions of θ annihilated by a single application of L. The eigenfunctions in $\mathscr{S}(\mathbb{R}^n)$ of θ annihilated by L are of the form Pg with P a harmonic, homogeneous polynomial. As discussed, the eigenfunctions of θ in $\mathscr{S}(\mathbb{R}^n)$ form an orthonormal basis of $L^2(\mathbb{R}^n)$.

Letting

$$\mathfrak{H}_d \otimes M_{n/2+d} = \{\mathbb{C}Pg \oplus \mathbb{C}R \cdot Pg \oplus \mathbb{C}R^2 \cdot Pg \oplus \cdots : P \in \mathfrak{H}^d\},$$

denote the $i(n/2+d)$-lowest weight space of $\mathrm{SO}(n) \times \mathfrak{sl}_2$, we have just shown that $L^2(\mathbb{R}^n)$ decomposes as

$$L^2(\mathbb{R}^n) = \text{ completion of } \bigoplus_{d \geq 0} \mathfrak{H}_d \otimes M_{n/2+d}.$$

Despite its merit as a decomposition into irreducible subrepresentations, this is not an decomposition into θ-eigenspaces. Rather, the $i(d+n/2)$ eigenspace of θ is the *cross section*

$$i(d+n/2)\text{-eigenspace of } \theta = \mathfrak{H}^d \cdot g \oplus R(\mathfrak{H}^{d-2} \cdot g) \oplus R^2(\mathfrak{H}^{d-4} \cdot g) \oplus \cdots \oplus R^d(\mathfrak{H}^0 \cdot g).$$

Consequently

> The eigenfunctions of θ form an orthonormal basis of $L^2(\mathbb{R}^n)$,
>
> $$L^2(\mathbb{R}^n) = \bigoplus_{d \geq 0} \left(\bigoplus_{a=0}^{\lceil d/2 \rceil} R^a(\mathfrak{H}^{d-2a} \cdot g) \right) = \bigoplus_{d \geq 0} (i(d+n/2)\text{-eigenspace of } \theta).$$

Chapter 4

Harmonic analysis on $L^2(\Gamma\backslash\mathfrak{H})$

4.0.4 Why the modular group Γ? Why the modular curve $\Gamma\backslash\mathfrak{H}$?

Depending on the reader's background, the quotient of the complex upper half plane $\mathfrak{H} = \{x + iy \in \mathbb{C} : y > 0\}$ by the modular group $\mathrm{SL}_2(\mathbb{Z})$, and functions thereon, may seem like peculiar objects to study. In this section, I give two motivations. The first invokes some facts from metric geometry, and enunciates how the study of functions on $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathfrak{H}$ introduces new spectral phenomena. The second motivation is based on the role of $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathfrak{H}$ as a *moduli space* of lattices in $\mathbb{C}$, thereby parametrizing complex tori, and in turn elliptic curves over $\mathbb{C}$.

Geometry As a first motivation, $\mathfrak{H}$ is a Riemann surface[1] with rich global geometric structure and serves as a model for the hyperbolic plane. Given a suitable metric that encodes this geometry, the Lie group $G = \mathrm{SL}_2(\mathbb{R})$ acts as *orientation–preserving isometries*, in the same way that the Lie group $(\mathbb{R}, +)$ acts by isometries of the one dimensional Euclidean space $\mathbb{R}$. Corresponding to this hyperbolic metric, $\mathfrak{H}$ is equipped with a G-invariant measure, and G-invariant Laplacian[2].

In studying functions on the noncompact space $\mathbb{R}$, we found that our analysis simplified when we looked at the compact *quotient*[3] $\mathbb{T} = \mathbb{Z} \setminus \mathbb{R}$. We found that $L^2(\mathbb{T})$ decomposed orthogonally into eigenspaces of the Euclidean Laplacian, despite none of those eigenfunctions being square integrable on $\mathbb{R}$ itself. Unlike $\mathbb{R}$ which has only one discrete subgroup $\mathbb{Z}$ (up to scale), the isometry group $G = \mathrm{SL}_2(\mathbb{R})$ has many nonisomorphic discrete subgroups H (see [2]), each giving rise to geometrically distinct quotients $H \setminus \mathfrak{H}$. Unlike $\mathbb{R}$ with its compact quotient $\mathbb{Z} \setminus \mathbb{R}$, the quotients $H \setminus \mathfrak{H}$ needn't be compact, depending on the subgroup H. For subgroups H such that $H \setminus \mathfrak{H}$ *is* compact[4], an argument analogous to that on the sphere

[1] that is, a space that has local analytic structure like $\mathbb{C}$

[2] Though, we will see that the invariant measure and Laplacian also arise naturally *group theoretically*, rather than *metric geometrically*, as presented here

[3] We write $\mathbb{Z}$ on the left to emphasize that $\mathbb{R}$ is the underlying *space*, and $\mathbb{Z}$ is a subgroup of the isometry group $\mathbb{R}$

[4] Such subgroups are frequently called *cocompact*

shows that the space $L^2(H \setminus \mathfrak{H})$ *does* have an orthonormal basis of eigenfunctions for the hyperbolic Laplacian. In this way, studying functions on compact quotients of $\mathfrak{H}$ introduces no new *spectral* phenomena.

However, there are discrete subgroups of G such as $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ that do not give rise to compact quotients. Rather, the space $\Gamma\backslash\mathfrak{H}$ is *unbounded*, due to the existence of a cusp (defined later). From our observation that $\Delta^{\mathbb{R}^n}$ has *no* eigenfunctions in $L^2(\mathbb{R}^n)$, we should be hesitant to believe that there are *any* square integrable eigenfunctions of the hyperbolic Laplacian on $\Gamma\backslash\mathfrak{H}$.

Yet, there *are* square integrable eigenfunctions of the hyperbolic Laplacian on $\Gamma\backslash\mathfrak{H}$. In fact, such eigenfunctions span a closed *infinite* dimensional subspace of $L^2(\Gamma\backslash\mathfrak{H})$ consisting of square integrable *cuspforms* (defined later). In contrast to the orthonormal basis of eigenfunctions (of the relevant operator) on tori, spheres, and $\mathbb{R}^n$, which we were able to write down explicitly, the square integrable eigenfunctions of the hyperbolic Laplacian are mysterious. Nonetheless we will show that the hyperbolic Laplacian has a self-adjoint extension (Friedrichs'), which, restricted to cuspforms, has compact resolvent (Rellich), with smooth, rapidly decaying eigenfunctions (Sobolev). Thus, despite explicit determination of such eigenfunctions being out of reach, the methods developed in this thesis show that they form an orthonormal basis of the space of square integrable cuspforms.

Moduli space [5]A lattice in $\mathbb{C}$ is a discrete additive subgroup L such that $L \otimes \mathbb{R} = \mathbb{C}$. A lattice can be specified in terms of generators: the lattice generated by two $\mathbb{R}$-linearly independent $z_1, z_2 \in \mathbb{C}$ is $\langle z_1, z_2\rangle = z_1\mathbb{Z} \oplus z_2\mathbb{Z}$. Any lattice L determines a *complex torus* $\mathbb{C}/L$, where the quotient is as additive groups. Since $\mathbb{C}$ is abelian, $\mathbb{C}/L$ is a group. Further, thinking locally, $\mathbb{C}/L$ is a Riemann surface, so one can define holomorphic and meromorphic maps on $\mathbb{C}/L$. Any L-periodic function $f : \mathbb{C} \to \mathbb{C}$ (that is, $f(z + \omega) = f(z)$ for all $\omega \in L$) factors through the torus $\mathbb{C}/L$, and conversely any map on $\mathbb{C}/L$ gives rise to one on $\mathbb{C}$ by periodicity. Since a holomorphic doubly periodic function is *bounded* and thus *constant* by Liouville, any holomorphic map on $\mathbb{C}/L$ is constant. However, there are nonconstant *meromorphic* maps on $\mathbb{C}/L$, an example of which is the *Wierstrass* function

$$\wp_L(z) = \frac{1}{z^2} + \sum_{\omega\in L-\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right) \qquad \text{for all } z \in \mathbb{C} - L.$$

The derivative of $\wp$ is also L periodic,

$$\wp_L'(z) = -2\sum_{\omega\in L} \frac{1}{(z-\omega)^3}.$$

Not-at-all-obviously, *all* meromorphic functions on $\mathbb{C}/L$ are ratios of polynomials in $\wp_L$ and $\wp_L'$. Again not obviously, $\wp_L$ and $\wp_L'$ satisfy a non-linear algebraic equation, with coefficients $g_2(L)$ and $g_4(L)$ depending on L

$$(\wp_L')^2 = 4(\wp_L)^3 - g_2(L)\wp_L - g_3(L).$$

[5]For a careful treatment of this material, see [5]

With $z \in \mathbb{C}/L$, the map $z \longmapsto (\wp_L, \wp'_L)$ parametrizes the solutions to the curve

$$Ell_L : y^2 = 4x^3 - g_2(L)x - g_3(L).$$

Not obviously, this parametrization is *bijective*, once one suitably accounts for points 'at infinity.' The cubic curve Ell_L is an *elliptic curve*, and this series of invocations has shown that every lattice in $\mathbb{C}$ gives rise to a complex torus, and that torus bijects to the solution set of an elliptic curve.

The complex-valued functions g_2 and g_3 of lattice-input are the holomorphic Eisenstein series, of weight 4 and 6 respectively. For now, we suppress details, and instead invoke that every elliptic curve $Ell_?$, defined by complex numbers a_2 and a_3 such that $a_2^3 - 27a_3^2 \neq 0$,

$$Ell_? : y^3 = 4x^3 - a_2x - a_3$$

actually arises from some lattice. That is, there is some lattice L such that $g_2(L) = a_2$ and $g_3(L) = a_3$, so that $Ell_? = Ell_L$.

Evidently, there is a bijection of *sets*

$$\text{complex tori} \leftrightsquigarrow \text{solution sets of elliptic curves over } \mathbb{C}.$$

Through this correspondence, the study of a complicated geometric and algebraically rich object (elliptic curve) becomes the study of a simple geometric and algebraic object (complex torus). Furthermore, the correspondence is more than just a set map, holomorphic group isomorphisms of complex tori correspond to coordinate transformations of elliptic curves. Thus, to classify elliptic curves up to coordinate transformations, it suffices to determine complex tori up to holomorphic group isomorphisms.

Initially the collection of complex tori is merely a set. It is preferable to parametrize the space of equivalence classes of complex tori by some topological or geometric object, so that analysis can be brought to bear on the classification of elliptic curves. This object is $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathfrak{H}$.

Complex tori are parametrized by the set of lattices in $\mathbb{C}$. For a given lattice L, there are many pairs $\{z_1, z_2\}$ of complex numbers such that $\langle z_1, z_2 \rangle = L$. A first parametrization of complex tori is then the collection of all $\mathbb{R}$-bases of $\mathbb{C}$, though this parametrization is not bijective. A first refinement is by taking *oriented* bases, (z_1, z_2) and requiring $z_1/z_2 \in \mathfrak{H}$. Since $z_1\mathbb{Z} \oplus z_2\mathbb{Z} = z_2\mathbb{Z} \oplus z_1\mathbb{Z}$, this does not eliminate any information. Even so, the parametrization by oriented bases is still not bijective. Rather two bases $\{z_1, z_2\}$ and $\{z'_1, z'_2\}$ generate the same lattice if and only if there is some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $(z_1, z_2)^\top = \gamma(z'_1, z'_2)^\top$ where γ acts as a linear transformation on the vector $(z'_1, z'_2)^\top$. Thus, there is a bijection

$$\mathrm{SL}_2(\mathbb{Z}) \backslash \text{oriented bases} \leftrightsquigarrow \text{complex tori}.$$

Two complex tori $\mathbb{C}/L$ and $\mathbb{C}/L'$ are holomorphically group isomorphic if and only if there is some complex number λ such that $L = \lambda L'$. We call two such lattices *homothetic*.

A defining feature of the lattice functions g_2 and g_3 appearing above, is that they are *homogeneous* with respect to homothety. That is, for any lattice L and $\lambda \in \mathbb{C}^\times$,

$$g_2(\lambda L) = \lambda^{-4} g_2(L) \quad \text{and} \quad g_3(\lambda L) = \lambda^{-6} g_3(L).$$

Consequently for two homothetic lattices L and λL, the corresponding elliptic curves transform

$$Ell_L : y^2 = 4x^3 - g_2(L)x - g_3(L) \quad \text{and} \quad Ell_{\lambda L} : y^2 = 4x^3 - \lambda^{-4} g_2(L)x - \lambda^{-6} g_3(L).$$

These elliptic curves are equivalent under the coordinate transformation $(x, y) \mapsto (\lambda^2 x, \lambda^3 y)$. Conversely, any such coordinate transformation arises from a homothety of the corresponding lattice. Thus, to parametrize the collection of elliptic curves up to coordinate transformations, we seek to parametrize lattices up to homothety.

For the lattice L generated by the ordered basis (z_1, z_2), the homothety $L \mapsto z_2^{-1} L$ distinguishes a representative of the form $L_\tau = \langle \tau, 1 \rangle$ where $\tau \in \mathfrak{H}$. Thus, the set of lattices up to homothety is parametrized by the complex upper half plane $\mathfrak{H}$. However we have already seen that a lattice distinguishes an ordered basis only up to translation by $\mathrm{SL}_2(\mathbb{Z})$. To make the parametrization of lattices up to homothety injective, we must determine how $\mathrm{SL}_2(\mathbb{Z})$ acts on representatives of the form $L_\tau = \tau\mathbb{Z} \oplus \mathbb{Z}$. Compute, using the action of $\mathrm{SL}_2(\mathbb{Z})$ on ordered bases

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \tau \\ 1 \end{bmatrix} = \begin{bmatrix} a\tau + b \\ c\tau + d \end{bmatrix} \quad \text{with } a, b, c, d \in \mathbb{Z}, \text{ and } ad - bc = 1.$$

Acting on the lattice $L_\tau = \langle a\tau + b, c\tau + d \rangle$ by the homothety $L_\tau \mapsto (c\tau + d)^{-1} L_\tau$ reveals the *fractional linear transformation* on $\mathfrak{H}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \tau \longmapsto \frac{a\tau + b}{c\tau + d}.$$

We *define* the action $\mathrm{SL}_2(\mathbb{Z})$ on $\mathfrak{H}$ to be that in the display above, enforcing compatibility with equivalence of lattices up to homothety.

Consequently any two points τ and τ' in $\mathfrak{H}$ generate the same lattice up to homothety, if and only if there is some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma\tau = \tau'$. Thus,

> *the topological structure parametrizing elliptic curves, up to coordinate change, is the quotient* $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathfrak{H}$.

In consequence, functions on $\mathfrak{H}$ that are $\mathrm{SL}_2(\mathbb{Z})$ invariant can be thought of as *invariants* of elliptic curves. The quotient $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathfrak{H}$ is richly structured: we can impose restrictions on these invariants, such as holomorphy, continuity, or square integrability, to name a few. In what follows, we will give a spectral decomposition of $L^2(\mathrm{SL}_2(\mathbb{Z}) \setminus \mathfrak{H})$, and in doing so we determine a spectral basis for the space of all square integrable invariants of elliptic curves, up to coordinate change.

4.0.5 $\mathfrak{H}$ as a homogeneous space

The entire complex plane $\mathbb{C}$ is acted on by $G = \mathrm{SL}_2(\mathbb{R})$ by linear fractional transformations

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times z \longmapsto \frac{az+b}{cz+d}.$$

The computation

$$\mathrm{Im}\,\frac{az+b}{cz+d} = \frac{\mathrm{Im}\,z}{|cz+d|^2}$$

shows that the complex upper half plane

$$\mathfrak{H} = \{z \in \mathbb{C} : \mathrm{Im}\,z > 0\}$$

is preserved under the action of $\mathrm{SL}_2(\mathbb{R})$. This action is *transitive*, since

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{bmatrix} \times i \longmapsto x + iy \qquad \text{for } x + iy \in \mathfrak{H}.$$

To compute the isotropy subgroup in $\mathrm{SL}_2(\mathbb{R})$ of i, observe

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} i = i \implies ai + b = di - c \implies a = d \text{ and } b = -c.$$

Since also $ad - bc = 1$, we see $a^2 + b^2 = c^2 + d^2 = 1$. That is,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SO}(2).$$

Moreover, letting $k_\theta = \left[\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right]$, we see

$$k_\theta \times i \longmapsto \frac{ie^{i\theta}}{e^{i\theta}} = i,$$

so the isotropy subgroup of i is

$$K = \mathrm{SO}(2).$$

Thus, there is an isomorphism of smooth G-spaces

$$G/K \approx \mathfrak{H} \quad \text{via} \quad gK \longmapsto gi.$$

We know that the matrix $n_x m_y = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{bmatrix}$ takes i to $x + iy$. On the other hand, if any other g takes i to $x + iy$, then $(n_x m_y)^{-1} g i = i$, so that $n_x m_y K = gK$. Thus, each coset gK has a unique representative of the form $n_x m_y K$ with $y > 0$.

Define subgroups of $G = \mathrm{SL}_2(\mathbb{R})$

$$P = \{\begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \in G\} \quad N = \{\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \in G\} \quad M = \{\begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \in G\},$$

where P is the semidirect product of M and N, with M normalizing N. The last paragraph proved the *Iwasawa decomposition*

$$G = \mathrm{SL}_2(\mathbb{R}) = PK = NMK.$$

In particular, we can parametrize G

$$\mathbb{R} \times \mathbb{R}_{>0} \times \mathbb{T} \to G \quad (x, y, \theta) \longmapsto n_x m_y k_\theta.$$

Remark 6. This parametrization shows that G is homeomorphic to $\mathfrak{H} \times \mathbb{T}$. Since $\mathbb{T}$ is compact (and further, abelian), the process of restriction to right-K invariant functions on G does not eliminate too much information. Once the invariant measure on G is discussed, *every* continuous function on G gives rise to a right-K invariant function by *averaging*. Furthermore, every continuous right-K invariant function is the average of some compactly supported continuous function on G. Thus, by studying functions on $\mathfrak{H} \approx G/K$, we are effectively studying functions on G itself.

4.0.6 Invariant operators on $\mathfrak{H}$

Invariant Laplacian

From the condition $\det g = 1$ for $g \in G$, and because $\det(e^X) = e^{\mathrm{tr}(X)}$, we see

$$e^X \in G \iff \mathrm{tr}(X) = 0.$$

Thus,

$$\mathfrak{g} = \mathfrak{sl}_2 = 3 \text{ dimensional real vectorspace of } 2 \times 2 \text{ matrices with trace } 0.$$

Take as a basis for $\mathfrak{g}$

$$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

satisfying the bracket relations

$$[H, X] = 2X \quad [H, Y] = -2Y \quad [X, Y] = H.$$

With respect to the nondegenerate pairing $\langle A, B \rangle = \mathrm{tr}(AB)$, the dual basis to $\{X, Y, H\}$ is

$$X^* = Y \quad Y^* = X \quad H^* = \frac{1}{2} H.$$

Thus, the Casimir element for $\mathfrak{g}$ is

$$\Omega^{\mathfrak{g}} = \frac{1}{2} H^2 + XY + YX.$$

This operator is designed to commute with the action of G: letting T_g be either the right or left regular representation, the Casimir element satisfies

$$\Omega^{\mathfrak{g}}(T_g f) = T_g(\Omega^{\mathfrak{g}} f) \qquad \text{for all } f : G \to \mathbb{C} \text{ and } g \in G$$

Consequently, $\Omega^{\mathfrak{g}}$ preserves right K invariance for functions on G, and thus descends to an operator for functions on $\mathfrak{H} \approx G/K$.

To determine the action of Ω on right K invariant functions $f : G \to \mathbb{C}$, compute exponentials

$$e^{tX} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} = n_t \in N, \quad e^{tH} = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} = m_{e^{2t}} \in M, \quad e^{tY} = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$$

Knowing that the subgroup $P = NM$ of G is a semidirect product, with M normalizing N, compute

$$m_y n_t = (m_y n_t m_y^{-1}) m_y = n_{yt} m_y.$$

Then compute, thinking of $n_x m_y = (x, y)$ as coordinates on G/K,

$$X \cdot f(n_x m_y) = \frac{d}{dt}\Big|_0 f(n_x m_y n_t) = \frac{d}{dt}\Big|_0 f(n_{x+yt} m_y) = \frac{d}{dt}\Big|_0 f(x + yt, y) = y \frac{\partial}{\partial x} f(n_x m_y)$$

and

$$H \cdot f(n_x m_y) = \frac{d}{dt}\Big|_0 f(n_x m_y m_{e^{2t}}) = \frac{d}{dt}\Big|_0 f(x, ye^{2t}) = 2y \frac{\partial}{\partial y} f(n_x m_y).$$

Rather than computing the action of Y on f directly, observe that the element $\theta = X - Y \in \mathfrak{g}$ exponentiates into K. Thus $e^{t\theta}$ fixes every right invariant function, so that $\theta \cdot f = (X - Y) \cdot f = 0$. Consequently, on a right K invariant function f, we have $X \cdot f = Y \cdot f$.

Let $\Delta^{\mathfrak{H}}$ denote the restriction of $\Omega^{\mathfrak{g}}$ to right K invariant functions $f : G \to \mathbb{C}$. Then, using the relation $YX = XY - H$, compute

$$\begin{aligned}
\Delta^{\mathfrak{H}} f &= (\frac{1}{2}H^2 + XY + YX) \cdot f = (\frac{1}{2}H^2 + 2XY - H) \cdot f = (\frac{1}{2}H^2 - H + 2X^2) \cdot f \\
&= (\frac{1}{2}(2y\frac{\partial}{\partial y})^2 - (2y\frac{\partial}{\partial y}) + 2(y\frac{\partial}{\partial x})^2) f = (2y^2(\frac{\partial}{\partial y})^2 + 2y\frac{\partial}{\partial y} - 2y\frac{\partial}{\partial y} + 2y^2(\frac{\partial}{\partial x})^2) f \\
&= 2y^2((\frac{\partial}{\partial x})^2 + (\frac{\partial}{\partial y})^2) f.
\end{aligned}$$

Renormalize so that

$$\Delta^{\mathfrak{H}} f = y^2(f_{xx} + f_{yy}).$$

By design, the operator $\Delta^{\mathfrak{H}}$ commutes with the right and left action of G on functions $f : G \to \mathbb{C}$. Consequently, $\Delta^{\mathfrak{H}}$ acts on functions $f : \mathfrak{H} \to \mathbb{C}$. We call $\Delta^{\mathfrak{H}}$ the **hyperbolic Laplacian**.

Take as the domain of $\Omega^{\mathfrak{g}}$

$$C_c^\infty(G) = \text{ smooth, compactly supported } f : G \to \mathbb{C}.$$

The subspace of right-K invariant functions in $C_c^\infty(G)$ is naturally identified with $C_c^\infty(\mathfrak{H})$, the complex vectorspace of smooth[6] compactly supported functions on $\mathfrak{H}$. Take $C_c^\infty(\mathfrak{H})$ as the domain of $\Delta^{\mathfrak{H}}$.

[6]Despite $\mathfrak{H}$ being a subset of $\mathbb{C}$, we do not take smooth to mean *complex analytic*, but rather smooth as a subset of $\mathbb{R}^2$

Invariant measure

As a locally compact group $G = \mathrm{SL}_2(\mathbb{R})$ admits a left-invariant *Haar* measure $\mathrm{d}g$. Define the Hilbert space,

$$L^2(G) = \text{ completion of } C^\infty(G) \text{ with respect to } |f|^2_{L^2(G)} = \int_G |f(g)|^2 \,\mathrm{d}g$$

As proved in [26], G is generated by commutators. Since the modular function Δ^G of $\mathrm{d}g$ is a homomorphism to the abelian group $\mathbb{R}^\times_{\geq 0}$, it is trivial on commutators. Consequently, G is unimodular, meaning that $\mathrm{d}g$ is also *right*-G invariant.

For $\Delta^{\mathfrak{H}}$ to be a symmetric unbounded operator on a Hilbert space, we want a left-G invariant measure on $\mathfrak{H}$. Since $\mathfrak{H} \approx G/K$, we determine the invariant measure on $\mathfrak{H}$ by determining a left-G invariant measure on G/K. In order for the invariant measure $\mathrm{d}g$ to descend to a unique (up to normalization) left-G invariant measure on G/K, the modular function Δ^G of G must restrict to the modular function Δ^K of $\mathrm{d}k$. That is, $\Delta^G\big|_K = \Delta^K$. Since Δ^G is trivial, it suffices to prove that Δ^K is trivial. Since K is *compact*, and since $\Delta^K : K \to \mathbb{R}^\times_{>0}$ is continuous, the image $\Delta^K(K)$ in $\mathbb{R}^\times_{>0}$ is compact. Since Δ^K is a homomorphism, its image is a subgroup of $R^\times_{>0}$. The only compact subgroup of $\mathbb{R}^\times_{>0}$ is the trivial subgroup. Consequently, Δ^K is trivial, as desired.

Thus, G/K is equipped with a left-G invariant measure $\mathrm{d}\overline{g}$, where $\overline{g} = gK$. The measure $\mathrm{d}\overline{g}$ is characterized by the formula

$$\int_G f(g)\,\mathrm{d}g = \int_{G/K}\int_K f(gk)\,\mathrm{d}k\,\mathrm{d}\overline{g} \qquad \text{for all } f \in C_c^\infty(G).$$

In particular, if we normalize $\mathrm{d}k$ so that K has measure 1, then for a right-K invariant function $f \in C_c^\infty(G)$

$$\int_G f(g)\,\mathrm{d}g = \int_{G/K}\int_K f(gk)\,\mathrm{d}k\,\mathrm{d}\overline{g} = \int_{G/K}\int_K f(g)\,\mathrm{d}k\,\mathrm{d}\overline{g} = \int_{G/K} f(\overline{g})\,\mathrm{d}\overline{g}.$$

Thus, to integrate $f \in C_c^\infty(\mathfrak{H})$ over $\mathfrak{H}$, we can integrate the corresponding right-K invariant function $g \longmapsto f(gi)$ on G. Provisionally, we denote the corresponding measure on $\mathfrak{H}$ by $\mathrm{d}\mu$, so that

$$\int_{\mathfrak{H}} f(\tau)\,\mathrm{d}\mu(\tau) = \int_{G/K} f(\overline{g}i)\,\mathrm{d}\overline{g} = \int_G f(gi)\,\mathrm{d}g.$$

The measure $\mathrm{d}\mu$ is left-G invariant and unique by design. For many purposes, mere existence and uniqueness of such a measure is sufficient, but to perform computations, we need an explicit formula for $\mathrm{d}\mu$.

The Iwasawa decomposition of G is

$$G = PK = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \cdot \mathrm{SO}(2).$$

The measure on G decomposes as a product of the left invariant measures $\mathrm{d}p$ on P and $\mathrm{d}k$ on K, in the sense that

$$\int_G f(g)\,\mathrm{d}g = \int_K \int_P f(pk)\,\mathrm{d}p\,\mathrm{d}k \quad \text{for all } f \in L^2(G).$$

On right-K invariant functions on G identified with $f \in C_c^\infty(\mathfrak{H})$, since K has volume 1, we see

$$\int_{\mathfrak{H}} f(\tau)\,\mathrm{d}\mu = \int_K \int_P f(pki)\,\mathrm{d}p\,\mathrm{d}k = \int_P f(pi)\,\mathrm{d}p.$$

Thus, to determine $\mathrm{d}\mu$ in terms of the coordinates of τ, it suffices to determine the measure $\mathrm{d}p$ on P in terms of coordinates of p.

Usefully the parabolic subgroup P has a *Levi* decomposition

$$P = NM = \{n_x m_y : n_x = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \in N, \quad m_y = \begin{bmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{bmatrix} \in M\}.$$

Thus, a priori, we know that there is a continuous function $\varphi : \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ such that

$$\int_P f(p)\,\mathrm{d}p = \int_{NM} f(n_x m_y)\,\mathrm{d}(n_x m_y) = \int_{\mathbb{R}} \int_{\mathbb{R}_{>0}^\times} f(n_x m_y)\varphi(x,y)\,\mathrm{d}^\times y\,\mathrm{d}^+ x,$$

where $\mathrm{d}^+ x$ is the additive Haar measure on $\mathbb{R}$ and and $\mathrm{d}^\times y = \frac{\mathrm{d}^+ y}{y}$ is the multiplicative Haar measure on $\mathbb{R}_{>0}^\times$. Since $\mathrm{d}p$ is left-P invariant, it is left-M and left-N invariant. Take $n_t \in N$, and compute using additive invariance of $\mathrm{d}^+ x$,

$$\begin{aligned} \int_P f(p)\,\mathrm{d}p = \int_P f(n_t p)\,\mathrm{d}p &= \int_{\mathbb{R}} \int_{\mathbb{R}_{>0}^\times} f(n_{x+t} m_y)\varphi(x,y)\,\mathrm{d}^\times y\,\mathrm{d}^+ x \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}_{\geq 0}^\times} f(n_x m_y)\varphi(x-t,y)\,\mathrm{d}^\times y\,\mathrm{d}^+ x. \end{aligned}$$

Thus φ is a function of y alone. Further, take $m_t \in M$ and compute using the commutation identity $m_t n_x = n_{tx} m_t$ and the multiplicative invariance of $\mathrm{d}^\times y$

$$\begin{aligned} \int_P f(p)\,\mathrm{d}p = \int_p (m_t p)\,\mathrm{d}p &= \int_{\mathbb{R}} \int_{\mathbb{R}_{>0}^\times} f(m_t n_x m_y)\varphi(y)\,\mathrm{d}^\times y\,\mathrm{d}^+ x \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}_{>0}^\times} f(n_{tx} m_{ty})\varphi(y)\,\mathrm{d}^\times y\,\mathrm{d}^+ x \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}_{>0}^\times} f(n_{tx} m_y)\varphi(y/t)\,\mathrm{d}^\times y\,\mathrm{d}^+ x. \end{aligned}$$

The change of variables $x \mapsto x/t$ results in $\mathrm{d}^+ x \mapsto 1/t\,\mathrm{d}^+ x$ so we need $\varphi(y/t) = t/y$. Consequently, $\varphi(y) = 1/y$.

Thus, the formula for the **hyperbolic measure** is

$$\mathrm{d}\mu(x+iy) = \frac{\mathrm{d}^+ x\,\mathrm{d}^\times y}{y} = \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.$$

Remark 7. The hyperbolic measure $\mathrm{d}x\,\mathrm{d}y/y^2$ reveals some essential geometry of $\mathfrak{H}$. For example, the majority of the volume of the upper half plane is concentrated near the boundary $y = 0$. Further, regions bounded away from $y = 0$, with x constrained to lie in some interval $[-L, L]$ have *finite volume*, despite possibly being unbounded as $iy \to i\infty$.

Functions on $\mathfrak{H}$

Recall that $C_c^\infty(\mathfrak{H})$ is the complex vectorspace of smooth, compactly supported functions on $\mathfrak{H}$. Define

$$L^2(\mathfrak{H}) = \text{ completion of } C_c^\infty(\mathfrak{H}) \text{ with respect to } |f|^2_{L^2(\mathfrak{H})} = \int_{\mathfrak{H}} |f(x+iy)|^2 \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.$$

By construction, $L^2(\mathfrak{H})$ is a Hilbert space, with inner product

$$\langle f, g\rangle = \int_{\mathfrak{H}} f(x+iy)\overline{g}(x+iy)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.$$

As with the sphere, the differential operators coming from the Lie algebra are *skew-symmetric* with respect to an invariant integral: compute for $A \in \mathfrak{sl}_2$ (now thinking of a function on $\mathfrak{H}$ as a right-K invariant function on G)

$$\int_{\mathfrak{H}} f(n_x m_y e^{tA})\overline{F}(n_x m_y)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} = \int_{\mathfrak{H}} f(n_x m_y)\overline{F}(n_x m_y e^{-tA})\frac{\mathrm{d}x\,\mathrm{d}y}{y^2},$$

then differentiating both sides at $t = 0$ shows

$$\langle Af, F\rangle = -\langle f, AF\rangle \quad \text{ for } f \in C^\infty(\mathfrak{H}).$$

Since $\Delta^{\mathfrak{H}} = \frac{1}{2}H^2 + XY + YX$, with each of H, X, and Y skew symmetric, we see that $\Delta^{\mathfrak{H}}$ is *symmetric*.

Moreover, since $X - Y$ acts by 0 on right-K invariant functions, compute

$$\langle \Delta^{\mathfrak{H}} f, f\rangle = -(\frac{1}{2}\langle Hf, Hf\rangle + \langle Yf, Xf\rangle + \langle Xf, Yf\rangle) = -(\frac{1}{2}|Hf|^2_{L^2(\mathfrak{H})} + 2|Xf|^2_{L^2(\mathfrak{H})}) \le 0.$$

Consequently, $-\Delta^{\mathfrak{H}}$ is *positive semi-definite*.

4.1 Functions on $\Gamma\backslash\mathfrak{H}$

The modular group Γ, the modular curve $\Gamma\backslash\mathfrak{H}$

As motivated in the opening to this chapter, we look at functions on $\Gamma\backslash\mathfrak{H}$, rather than $\mathfrak{H}$ itself. While we generally prefer to work intrinsically, characterizing the integral on $\Gamma\backslash\mathfrak{H}$ as the unique distribution satisfying

$$\int_{\mathfrak{H}} f(\tau)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} = \int_{\Gamma\backslash\mathfrak{H}} \left(\sum_{\gamma\in\Gamma} f(\gamma\tau)\right)\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \quad \text{ for all } f \in C_c^\infty(\mathfrak{H}),$$

it is sometimes useful to consider the **fundamental domain** for Γ:

$$\mathcal{F} = \{\tau \in \mathfrak{H} : \operatorname{Re}\tau \le 1/2,\ |\tau| \ge 1\}/\sim .$$

Here $\sim$ denotes some boundary identifications that occur on a set of measure zero. Integration of functions on $\Gamma\backslash\mathfrak{H}$ can be thought of in terms of the *unwinding* property in the first display, or by integration over representatives in the fundamental domain, and both methods are useful in different scenarios. For example, it is apparent that $\Gamma\backslash\mathfrak{H}$ has finite hyperbolic volume by explicitly evaluating the integral

$$\int_{\mathcal{F}} \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} = \int_{x=-1/2}^{1/2}\int_{y=\sqrt{1-x^2}}^{\infty} \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} = \int_{x=-1/2}^{1/2} (1-x^2)^{-1/2}\,\mathrm{d}x = \pi/3.$$

Descending to the quotient, Friedrichs' extension

Define the complex vectorspace

$$C_c^\infty(\Gamma\backslash\mathfrak{H}) = \text{ smooth, compactly supported, left-}\Gamma\text{-invariant } f:\mathfrak{H}\to\mathbb{C}.$$

Define the Hilbert space

$$L^2(\Gamma\backslash\mathfrak{H}) = \text{ completion of } C_b^\infty(\Gamma\backslash\mathfrak{H}) \text{ with respect to } |f|^2_{L^2(\Gamma\backslash\mathfrak{H})} = \int_{\Gamma\backslash\mathfrak{H}} |f(x+iy)|^2 \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

Since $\Delta^{\mathfrak{H}}$ is Γ invariant, it descends to an operator on $C_b^\infty(\Gamma\backslash\mathfrak{H})$. We have observed that $\Delta^{\mathfrak{H}}$ is a *symmetric*, *negative semi-definite* unbounded operator on $L^2(\mathfrak{H})$, and consequently satisfies the same properties as an unbounded operator on $L^2(\Gamma\backslash\mathfrak{H})$.

By Friedrichs' construction, $\Delta^{\mathfrak{H}}$ has a negative semidefinite *self-adjoint* extension $\widetilde{\Delta^{\mathfrak{H}}}$ defined on a dense subspace of the (automorphic) Sobolev space

$$H^1(\Gamma\backslash\mathfrak{H}) = \text{ completion of } C_c^\infty(\Gamma\backslash\mathfrak{H}) \text{ with respect to } |f|^2_{L^2(\Gamma\backslash\mathfrak{H})} = \langle(\lambda - \widetilde{\Delta^{\mathfrak{H}}})f, f\rangle.$$

The extension $\widetilde{\Delta^{\mathfrak{H}}}$ is characterized by the relation on its resolvent $(\lambda - \widetilde{\Delta^{\mathfrak{H}}})^{-1}$, a bounded operator $L^2(\Gamma\backslash\mathfrak{H}) \to H^1(\Gamma\backslash\mathfrak{H}) \subset L^2(\Gamma\backslash\mathfrak{H})$,

$$\langle f, g\rangle = \langle(1-\Delta^{\mathfrak{H}})f, (\lambda - \widetilde{\Delta^{\mathfrak{H}}})^{-1}g\rangle \quad \text{for all } f\in C_c^\infty(\Gamma\backslash\mathfrak{H}) \text{ and } g \in L^2(\Gamma\backslash\mathfrak{H}).$$

As usual, to prove that $(\lambda - \widetilde{\Delta^{\mathfrak{H}}})^{-1} : L^2(\Gamma\backslash\mathfrak{H}) \to L^2(\Gamma\backslash\mathfrak{H})$ is compact, it suffices to prove that the inclusion $H^1(\Gamma\backslash\mathfrak{H}) \to L^2(\Gamma\backslash\mathfrak{H})$ is compact.

In the previous chapters the routine of invoking Friedrichs construction was followed immediately by a proof of the desired compactness. Then upon invoking the spectral theorem for compact self-adjoint operators, we obtain an orthonormal basis of eigenfunctions for the resolvent of the extension, in turn of the extension itself. Last, a proof of a relevant Sobolev regularity showed that eigenfunctions of the extension must be in the domain of the original operator, whereat the extension and the original agree.

For reasons that will become clear soon, this will fail:

The inclusion $H^1(\Gamma\backslash\mathfrak{H}) \to L^2(\Gamma\backslash\mathfrak{H})$ is not compact.

This failure is due to the existence of some *continuous* spectrum, corresponding to functions analogous to the *not* square integrable functions $\xi \to e^{i\langle x,\xi\rangle}$ on $\mathbb{R}^n$, being eigenfunctions in a larger space. Nonetheless, there is a closed, infinite dimensional subspace of $H^1(\Gamma\backslash\mathfrak{H})$ which *does* embed compactly into $L^2(\Gamma\backslash\mathfrak{H})$. This is the space of *cuspforms*, characterized as the orthogonal complement to the *pseudo-Eisenstein series.* We postpone further discussion of spectral theory until we have constructed a suitable collection of explicit functions on $\Gamma\backslash\mathfrak{H}$.

4.1.1 Explicit functions: nonholomorphic Eisenstein series

The hyperbolic Laplacian is

$$\Delta^{\mathfrak{H}} = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

For any complex s, the function $f_s : \tau \mapsto (\operatorname{Im}\tau)^s$ is an $\lambda_s = s(s-1)$-eigenfunction of $\Delta^{\mathfrak{H}}$:

$$\Delta^{\mathfrak{H}} f(x+iy) = y^2\frac{\partial^2}{\partial y^2}y^s = y^2 s(s-1)y^{s-2} = s(s-1)y^s.$$

The function f_s is *not* Γ invariant. Rather,

$$f_s\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\tau\right) = \left(\operatorname{Im}\left(\frac{a\tau+b}{c\tau+d}\right)\right)^s = \frac{(\operatorname{Im}\tau)^s}{|c\tau+d|^{2s}} \quad \text{for } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma.$$

However, if $c = 0$ in the display above (forcing $a = d = 1$), then indeed $f_s(\tau + b) = f_s(\tau)$. Such matrices are in $P \cap \Gamma$, where P is the parabolic subgroup of upper triangular matrices. Thus, to enforce Γ-invariance, we take s in the right half plane $\operatorname{Re} s > 1$ and define the sth (nonholomorphic) **Eisenstein series**

$$E_s(\tau) = \sum_{\gamma\in P\cap\Gamma\backslash\Gamma} f_s(\gamma\tau) = \sum_{\gamma\in P\cap\Gamma\backslash\Gamma} \operatorname{Im}(\gamma\tau)^s.$$

For a fixed s with $\operatorname{Re} s > 1$, the sum defining E_s is uniformly convergent on compact sets in $\Gamma\backslash\mathfrak{H}$: working on the fundamental domain $\mathcal{F}$ where $-1/2 \le \operatorname{Re}(\tau) \le 1/2$ and $|\tau| > 1$, fix a compact subset bounded vertically by some constant iM, and observe for such $\tau = x + iy$,

$$\left|\begin{bmatrix} a & b \\ c & d \end{bmatrix}\operatorname{Im}(\tau)^s\right| = \left|\frac{y^s}{|c(x+iy)+d|^{2s}}\right| = \left(\frac{y}{(cx+d)^2+(cy)^2}\right)^{\operatorname{Re} s} = \left|\frac{y}{c^2(x^2+y^2)+2xcd+d^2}\right|^{\operatorname{Re} s}$$
$$\le \left|\frac{y}{c^2-cd+d^2}\right|^{\operatorname{Re} s} \le \left(\frac{M}{(c^2+d^2)}\right)^{\operatorname{Re} s}.$$

Consequently, for τ in that compact $E_s(\tau)$ is uniformly bounded

$$|E_s(\tau)| \le M^{\operatorname{Re}(s)} \sum_{(c,d)\ne(0,0)} (c^2+d^2)^{-\operatorname{Re} s}.$$

The latter sum is convergent for $\operatorname{Re} s > 1$, as claimed. Conversely, when $\operatorname{Re} s \le 1$, the sum defining E_s *diverges.*

By design, the Eisenstein series is a $\lambda_s = s(s-1)$-eigenfunction of $\Delta^{\mathfrak{H}}$, now thought of as a function on $\Gamma\backslash\mathfrak{H}$. There is an immediate conflict. Knowing that $\Delta^{\mathfrak{H}}$ is a symmetric, negative semi-definite operator, the eigenvalue $s(s-1)$ should be *real* and *non-positive*. An easy computation shows that this is only possible when $\operatorname{Re} s = 1/2$, or $s \in [0,1]$. But the Eisenstein series, defined *as a series*, is *not* convergent for any such s. Evidently, when $\operatorname{Re} s > 1$, the Eisenstein series E_s is *not* square integrable.

Meromorphic continuation of $s \longmapsto E_s(\tau)$

While for fixed s, the function $\tau \mapsto E_s(\tau)$ is *not* complex analytic, the function $s \mapsto E_s(\tau)$ (for fixed τ) *is*. Consequently, if we can make sense of E_s for $s(s-1)$ real and nonpositive, the function $\tau \mapsto E_s(\tau)$ will remain a $\Delta^{\mathfrak{H}}$ eigenfunction, by the identity principle of complex analysis.

In this section, we continue $s \mapsto E_s(\tau)$ (for fixed τ) to a meromorphic function on all of $\mathbb{C}$, show that the meromorphic continuation satisfies a functional equation, and show that for *any* fixed s the function $\tau \mapsto E_s(\tau)$ is of *moderate growth* in $\operatorname{Re}\tau$.

Recall the Euler-Riemann zeta, and its completion

$$\zeta(s) = \sum_{n>0} n^{-s} \qquad \operatorname{Re} s > 1$$

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) \qquad \operatorname{Re} s > 1.$$

We invoke that $\xi(s)$ has an analytic continuation to an entire function, satisfying the functional equation $\xi(s) = \xi(1-s)$ for all $s \in \mathbb{C}$.

Claim 10. Three claims,

- Fix $\tau \in \mathfrak{H}$. The function

  $$s \longmapsto s(1-s)\xi(2s)E_s(\tau)$$

  has an analytic continuation to an entire function of s. The only pole of $E_s(\tau)$ is at $s = 1$, and has residue $3/\pi$, independent of τ.

- Fix τ. The continuation satisfies

  $$\xi(2s)E_s(\tau) = \xi(2-2s)E_{1-s}(\tau) \quad \text{for all } s \in \mathbb{C}.$$

- Fix $s \in \mathbb{C}$. There is some constant A (dependent on s) such that

  $$|E_s(x+iy)| \leq y^A \quad \text{as } y \to \infty$$

Proof. **Analytic continuation:** Fix $\tau \in \mathfrak{H}$. To determine the continuation, we find an integral expression for $s \mapsto s(1-s)\xi(2s)E_s(\tau)$, that is nicely convergent for all $s \in \mathbb{C}$. Introduce the *theta function* on $\mathrm{SL}_2(\mathbb{R})$, writing $v \in \mathbb{Z}^2$ as a row vector,

$$\Theta(g) = \sum_{v \in \mathbb{Z}^2} e^{-\pi|vg|^2} \quad \text{for } g \in \mathrm{SL}_2(\mathbb{R}).$$

Letting $t \in \mathbb{R}_{>0}$ act on matrices by scaling and taking s with $\operatorname{Re} s > 1$, consider the *Mellin transform* of $t \longmapsto \Theta(tg)$, first integrating term by term using the rapid convergence of Θ,

$$I_{s,g} = \int_{\mathbb{R}_{>0}} t^{2s}(\Theta(tg) - 1)\frac{\mathrm{d}t}{t} = \sum_{v \neq (0,0)} \int_{\mathbb{R}_{>0}} t^{2s} e^{-\pi|tvg|^2} \frac{\mathrm{d}t}{t}.$$

Changing variables $t \longmapsto t/(\sqrt{\pi}|vg|)$, and then $t \to \sqrt{t}$,

$$I_{s,g} = \sum_{v \neq (0,0)} (\sqrt{\pi}|vg|)^{-2s} \int_{\mathbb{R}_{>0}} t^{2s} e^{-t^2} \frac{\mathrm{d}t}{t} = \frac{1}{2}\pi^{-s} \sum_{v \neq (0,0)} |vg|^{-2s}\Gamma(s).$$

Recognizing the sum over a lattice (with the origin excluded), we choose g to transform $\mathbb{Z}^2$ into the lattice generated by $\tau = x+iy$. This corresponds to evaluation of $I_{s,g}$ at $g = n_x m_y$. Compute for $v = (c,d) \in \mathbb{Z}^2 - \{0\}$,

$$\begin{aligned} |(c,d)n_x m_y|^{-2s} &= |(c, cx+d)m_y|^{-2s} = y^s|(cy)^2 + (cx+d)^2|^{-s} \\ &= \frac{y^s}{|c\tau + d|^{2s}} = \operatorname{Im}(\begin{bmatrix} * & * \\ c & d \end{bmatrix} \tau)^s. \end{aligned}$$

Thus, with a factor of $2\zeta(2s)$ accounting for the sum over the whole lattice rather than just points with coprime coordinates, we find

$$I_{s,n_x m_y} = \pi^{-s}\Gamma(s)\zeta(2s)E_s(\tau) = \xi(2s)E_s(\tau).$$

Second, observe that (with $\tau = x+iy$ and $g_\tau = n_x m_y$ fixed) the integral defining I_{s,g_τ} splits into two parts:

$$I_{s,g_\tau} = \left(\int_1^\infty + \int_0^1\right) t^{2s}(\Theta(tg_\tau) - 1)\frac{\mathrm{d}t}{t}.$$

The integral from 1 to ∞ converges absolutely (seen by dominating Θ with a geometric series), and is thus entire as a function of s. Initially, the convergence of the integral from 0 to 1 is fragile at the former endpoint. We convert the integral from 0 to 1 to an integral from 1 to ∞ by means of the transformation law (letting g' denote $(g^\top)^{-1}$)

$$t^2\Theta(t^{-1}g'_\tau) = \Theta(tg_\tau) \quad \text{for all } t > 0.$$

Granting[7] this display, the integral from 0 to 1 transforms as

$$\int_0^1 t^{2s}(\Theta(tg_\tau) - 1)\frac{\mathrm{d}t}{t} = \int_0^1 t^{2s}[t^{-2}(\Theta(t^{-1}g'_\tau) - 1) + t^{-2} - 1]\frac{\mathrm{d}t}{t}.$$

Change variables $t \to t^{-1}$, changing the integral from 0 to 1 to an integral from 1 to ∞,

$$\int_0^1 t^{2s}(\Theta(tg_\tau) - 1)\frac{\mathrm{d}t}{t} = \int_1^\infty t^{-2s}[t^2(\Theta(tg'_\tau) - 1) + t^2 - 1]\frac{\mathrm{d}t}{t}.$$

[7]Proof: the Fourier transform of $v \mapsto e^{-\pi|vtg|^2}$ is $v \mapsto t^{-2}e^{-\pi|vt^{-1}g'|^2}$. Then Poisson summation gives the display.

Since $\operatorname{Re} s > 1$, we can evaluate the integral of latter two summands, giving

$$\int_0^1 t^{2s}(\Theta(tg_\tau) - 1)\frac{\mathrm{d}t}{t} = \int_1^\infty t^{-2s+2}(\Theta(tg'_\tau) - 1)\frac{\mathrm{d}t}{t} + \frac{1}{2s-2} - \frac{1}{2s}.$$

Last, note that map $g \mapsto g' = (g^\top)^{-1}$ is equivalently the map $g \mapsto wgw^{-1}$ where

$$w = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in K = \mathrm{SO}(2).$$

Consequently, because w acts on $\mathbb{Z}^2$ as a rotation, the summands in $\Theta(g') = \Theta(wgw^{-1})$ are a permutation of the summands in $\Theta(g)$. Since Θ is absolutely convergent, permuting the summands does not effect the value of the sum. Consequently, $\Theta(g'_\tau) = \Theta(g_\tau)$. Consequently, for $\operatorname{Re} s > 1$.

$$\int_0^1 t^{2s}(\Theta(tg_\tau) - 1)\frac{\mathrm{d}t}{t} = \int_1^\infty t^{-2s+2}(\Theta(tg_\tau) - 1)\frac{\mathrm{d}t}{t} + \frac{1}{2s-2} - \frac{1}{2s}.$$

The integral on the right hand side is uniformly convergent for all $s \in \mathbb{C}$, and thus defines an entire function.

Putting this together with the first computation, initially only for $\operatorname{Re}(s) > 1$, we have

$$\xi(2s)E_s(\tau) = \int_1^\infty (t^{2s} + t^{2-2s})(\Theta(tg_\tau) - 1)\frac{\mathrm{d}t}{t} + \frac{1}{2s-2} - \frac{1}{2s}.$$

The right hand side is a meromorphic function of s, defined on all of $s \in \mathbb{C}$ with simple poles at $s = 0$ and $s = 1$. Thus, as claimed $s(s-1)\xi(2s)E_s(\tau)$ is entire.

Functional equation: The integral expression for $\xi(2s)E_s(\tau)$ is stable under the transformation $s \longmapsto 1 - s$, and the completed Euler-Riemann zeta function satisfies $\xi(2s) = \xi(2-2s)$. Consequently, as claimed

$$\xi(2s)E_s(\tau) = \xi(2-2s)E_{1-s}(\tau) \quad \text{for all } s \in \mathbb{C}.$$

Moderate growth (sketch) For $\operatorname{Re}(s) > 1$, we have already demonstrated moderate growth in y. From the functional equation, and the asymptotics of Γ, we have moderate growth in y for $\operatorname{Re}(s) < 0$. Then since E_s is easily dominated by some exponential $e^{N|s|}$, and since E_s is bounded (in s) on the vertical lines $\operatorname{Re}(s) = -\delta$ and $\operatorname{Re}(s) = 1 + \delta$, Phragmén–Lindelöf demonstrates moderate growth in y for all $s \in \mathbb{C}$, away from poles. For details, see [25]. □

Remark 8. While $\tau \in \mathfrak{H}$ was fixed throughout the analytic continuation, the integral expression

$$\xi(2s)E_s(\tau) = \int_1^\infty (t^{2s} + t^{2-2s})(\Theta(tg_\tau) - 1)\frac{\mathrm{d}t}{t} + \frac{1}{2s-2} - \frac{1}{2s}$$

is uniformly convergent for g_τ in a fixed compact set in $\mathrm{SL}_2(\mathbb{R})$. Consequently, for s not necessarily in the original region of convergence, the function $\tau \mapsto E_s(\tau)$ remains smooth. Thus, despite E_s not being bounded, we can still sensibly apply $\Delta^{\mathfrak{H}}$. This is analogous to $\Delta^{\mathbb{R}^n}$ acting on $\xi \mapsto e^{i\langle x,\xi\rangle}$ despite the latter not being square integrable.

Eisenstein series: Fourier component $n = 0$

Since E_s is Γ invariant, it is $\mathbb{Z}$-periodic in x. Being smooth and of moderate growth, the Eisenstein series E_s admits a Fourier expansion with Fourier *components* $C_n(y)$, dependent on y,

$$E_s(x+iy) = \sum_{n\in\mathbb{Z}-\{0\}} C_{n,s}(y)e^{2\pi inx} + C_{0,s}(y).$$

The nth Fourier component of E_s is

$$C_{n,s}(y) = \int_0^1 E_s(x+iy)e^{-2\pi inx}\,\mathrm{d}x.$$

In particular, the *constant term* c_PE_s is

$$c_PE_s(y) = \int_0^1 E_s(x+iy)\,\mathrm{d}x.$$

Multiplying E_s by $2\zeta(2s)$ removes the coprimality condition on the summands defining E_s in the initial region of convergence. When $\operatorname{Re} s > 1$, compute directly, exchanging the integral and the sum using absolute convergence,

$$2\zeta(2s)c_PE_s(y) = 2\zeta(2s)\int_0^1 E_s(x+iy)\,\mathrm{d}x = \sum_{(c,d)\neq(0,0)} \int_0^1 \frac{y^s}{|c(x+iy)+d|^{2s}}\,\mathrm{d}x.$$

Next, breaking up the sum into $\mathbb{Z}$-invariant terms, then unwinding to an integral over $\mathbb{R}$, compute (recognizing a factor of $2\zeta(s)$ in the first summand)

$$\begin{aligned} 2\zeta(2s)c_PE_s(y) &= \sum_{d\neq 0}\left(\int_0^1 \frac{y^s}{d^{2s}}\,\mathrm{d}x + y^s\sum_{c\neq 0}\int_0^1 \frac{1}{c^{2s}|x+iy+\frac{d}{c}|^{2s}}\,\mathrm{d}x\right)\\ &= 2\zeta(2s)y^s + y^s\sum_{c\neq 0}\frac{1}{c^{2s}}\sum_{d=0}^{c-1}\sum_{\ell\in\mathbb{Z}}\int_0^1 \frac{1}{|x+\ell+iy+\frac{d}{c}|^{2s}}\,\mathrm{d}x\\ &= 2\zeta(2s)y^s + y^s\sum_{c\neq 0}\frac{1}{c^{2s}}\sum_{d=0}^{c-1}\int_{\mathbb{R}} \frac{1}{|x+iy+\frac{d}{c}|^{2s}}\,\mathrm{d}x. \end{aligned}$$

Change variables $x \mapsto x+d/c$, a total of $|c|$ times for each $c \neq 0$, using translation invariance of $\mathrm{d}x$, we find

$$2\zeta(2s)c_PE_s(y) = 2\zeta(2s)y^s + y^s\sum_{c\neq 0}\frac{1}{c^{2s-1}}\int_{\mathbb{R}}\frac{1}{(x^2+y^2)^s}\,\mathrm{d}x.$$

Then change variables $x \mapsto xy$, recognizing a factor of $2\zeta(2s-1)$,

$$2\zeta(2s)c_PE_s(y) = 2\zeta(2s)y^s + 2\zeta(2s-1)y^{1-s}\int_{\mathbb{R}}\frac{1}{(x^2+1)^s}\,\mathrm{d}x.$$

The latter integral can be computed using the 'Gamma function trick,'

$$(x^2+1)^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{t(x^2+1)} t^s \frac{\mathrm{d}t}{t}.$$

Then compute,

$$2\zeta(2s) c_P E_s(y) = 2\zeta(2s) y^s + 2\zeta(2s-1) y^{1-s} \int_{\mathbb{R}} \frac{1}{\Gamma(s)} \int_0^\infty e^{t(x^2+1)} t^s \frac{\mathrm{d}t}{t} \,\mathrm{d}x.$$

Change variables $x \mapsto (\pi/t)^{1/2} x$, then

$$\begin{aligned} 2\zeta(2s) c_P E_s(y) &= 2\zeta(2s) y^s + 2\zeta(2s-1) y^{1-s} \int_{\mathbb{R}} \frac{\sqrt{\pi}}{\Gamma(s)} \int_0^\infty e^{-\pi x^2 + t} t^{s-1/2} \frac{\mathrm{d}t}{t} \,\mathrm{d}x \\ &= 2\zeta(2s) y^s + 2\zeta(2s-1) y^{1-s} \frac{\sqrt{\pi}}{\Gamma(s)} \int_0^\infty e^t t^{s-1/2} \frac{\mathrm{d}t}{t} \\ &= 2\zeta(2s) y^s + 2\zeta(2s-1) y^{1-s} \frac{\Gamma(s-1/2)\sqrt{\pi}}{\Gamma(s)}. \end{aligned}$$

Last, recalling $\xi(s) = \pi^{s/2}\Gamma(s/2)\zeta(s)$, we have found

$$c_P E_s(y) = y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s}.$$

Since this computation made use of the series characterization for E_s, this expression for the constant term is initially valid only for $\operatorname{Re} s > 1$. By the identity principal from complex analysis, for each y, we can continue the meromorphic function $s \mapsto c_P E_s(y)$ to all $s \in \mathbb{C}$.

Remark 9. After computing the nonzero Fourier components of E_s, we will see that the constant term $c_P E_s$ is responsible for the moderate growth (and thereby non-square integrability) of the Eisenstein series.

Eisenstein series: Fourier components $n \neq 0$, asymptotics

The explicit nature of E_s allows explicit computation of the Fourier components for $n \neq 0$. Before doing so, we examine the asymptotic features of the components as $y \to \infty$. We will see that the asymptotics of the solutions rely only on E_s being an eigenfunction of $\Delta^{\mathfrak{H}}$. Consequently, the discussion will apply to generic eigenfunctions satisfying moderate growth conditions, which will be useful in discussing *cuspforms*, whose Fourier components cannot be computed explicitly.

Since E_s is a $\lambda_s = s(s-1)$ eigenfunction of $\Delta^{\mathfrak{H}}$, differentiate termwise (invoking smoothness), and equate nonzero coefficients to obtain a family of ordinary differential equations,

$$y^2 C''_{n,s}(y) - (4\pi^2 n^2 y^2 + \lambda_s) C_{n,s} = 0. \tag{4.1}$$

Moreover, these differential equations are constrained by requiring *moderate growth* as $y \to \infty$.

Granting a solution $u = C_{1,s}$ to the equation corresponding to $n = 1$, the function $u_n(y) = u_1(|n|y)$ (for generic $n \neq 0$) satisfies

$$\begin{aligned} y^2 u_n''(y) - (4\pi^2 y^2 n^2 + \lambda_s) u_n(y) &= y^2 n^2 u_1''(|n|y) - (4\pi y^2 n^2 + \lambda_s) u_1(|n|y) \\ &= z^2 u_1''(z) - (4\pi z^2 + \lambda_s) u_1(z) = 0, \end{aligned}$$

showing that u_n (for $n \neq 0$) is a solution to the differential equation characterizing the Fourier component $C_{n,s}$. To determine the behavior of the nth component, it suffices to determine that of u_1.

Dividing through by y^2, we are looking for moderate growth solutions to the differential equation

$$u'' - (4\pi^2 + \frac{\lambda_s}{y^2})u = 0. \tag{4.2}$$

This differential equation has an *irregular singular point* at $y = \infty$. Such matters are treated in [13]. For such equations, we expect that there are solutions that behave *asymptotically* like solutions to the equation obtained by setting the coefficient function at $y = \infty$. That is, a solution to the equation in the display above will behave asymptotically like a solution to

$$u'' - 4\pi^2 u = 0.$$

This equation is easily solved, yielding $u_\pm(y) = e^{\pm 2\pi y}$. Of these two possibilities, only $u_-(y) = e^{-2\pi y}$ is of *moderate growth* (in fact, it decays *rapidly*). Thus, of the two possible solutions to equation 4.2, the moderate growth condition on E_s ensures that the solution u will decay rapidly as $y \to \infty$. We will find an integral expression for this solution by explicitly computing the Fourier components of E_s.

Eisenstein series: Fourier components $n \neq 0$, computation

Take $\operatorname{Re} s > 1$. It is easier to work with $2\zeta(2s)E_s$ than E_s itself, since the former is a sum over a whole lattice with $(0,0)$ removed. We want to compute

$$2\zeta(2s)C_{n,s}(y) = \int_0^1 \sum_{(c,d)\neq(0,0)} \frac{y^s}{c(x+iy)+d} e^{2\pi i n x}\,\mathrm{d}x.$$

As with the computation of the constant term, break the sum up into terms $\mathbb{Z}$-invariant in x. That is,

$$2\zeta(2s)C_{n,s}(y) = y^s \sum_{c\neq 0} \frac{1}{|c|^{2s}} \sum_{d=0}^{c-1} \int_0^1 \sum_{\ell\in\mathbb{Z}} \frac{1}{|x+\ell+\frac{d}{c}+iy|^{2s}} e^{-2\pi i n x}\,\mathrm{d}x.$$

The integral *unwinds*

$$2\zeta(2s)C_{n,s}(y) = y^s \sum_{c\neq 0} \frac{1}{|c|^{2s}} \sum_{d=0}^{c-1} \int_{\mathbb{R}} \frac{1}{|x+\frac{d}{c}+iy|^{2s}} e^{-2\pi i n x}\,\mathrm{d}x.$$

Changing variables $x \mapsto x + d/c$ a total of $|c|$ times, we have

$$2\zeta(2s)C_{n,s}(y) = y^s \sum_{c\neq 0} \frac{1}{|c|^{2s}} \sum_{d=0}^{c-1} e^{2\pi ind/c} \int_{\mathbb{R}} \frac{1}{|x+iy|^{2s}} e^{-2\pi inx}\,\mathrm{d}x.$$

The function $d \mapsto e^{2\pi ind/c}$ is a character of $\mathbb{Z}/c\mathbb{Z}$, and the sum over d is equivalently a sum over $\mathbb{Z}/c\mathbb{Z}$. The character is nontrivial if and only if c does not divide n. These terms vanish, by orthogonality. If c does divide n, then the sum evaluates to $|c| = |\mathbb{Z}/c\mathbb{Z}|$. Thus, letting $\sigma_s(n)$ be the sum of the sth powers of the divisors of n,

$$2\zeta(2s)C_{n,s}(y) = y^s\sigma_{2s-1}(n) \int_{\mathbb{R}} \frac{1}{|x+iy|^{2s}} e^{-2\pi inx}\,\mathrm{d}x.$$

Changing variables $x \mapsto xy$, then pulling out a factor of y^{2s},

$$\begin{aligned} 2\zeta(2s)C_{n,s}(y) &= y^{1-s}\sigma_{2s-1}(n) \int_{\mathbb{R}} \frac{1}{|x+i|^{2s}} e^{-2\pi inxy}\,\mathrm{d}x \\ &= y^{1-s}\sigma_{2s-1}(n) \int_{\mathbb{R}} \frac{1}{(x^2+1)^s} e^{-2\pi inxy}\,\mathrm{d}x. \end{aligned}$$

Using the 'Gamma function trick,' this is

$$\begin{aligned} 2\zeta(2s)C_{n,s}(y) &= y^{1-s}\sigma_{2s-1}(n) \int_{\mathbb{R}} \frac{1}{\Gamma(s)} e^{-2\pi inxy} \int_0^\infty t^s e^{-t(x^2+1)} \frac{\mathrm{d}t}{t}\,\mathrm{d}x \\ &= y^{1-s}\sigma_{2s-1}(n) \int_0^\infty \frac{1}{\Gamma(s)} t^s e^{-t} \int_{\mathbb{R}} e^{-2\pi inxy} e^{-tx^2}\,\mathrm{d}x \frac{\mathrm{d}t}{t}. \end{aligned}$$

Change variables $x \mapsto \sqrt{\pi/t}x$, and recognize the Fourier transform of the Gaussian at $ny\sqrt{\pi/t}$,

$$\begin{aligned} 2\zeta(2s)C_{n,s}(y) &= y^{1-s}\sigma_{2s-1}(n) \int_0^\infty \frac{\sqrt{\pi}}{\Gamma(s)} t^{s-1/2} e^{-t} \int_{\mathbb{R}} e^{-2\pi inxy\sqrt{\pi/t}} e^{-tx^2}\,\mathrm{d}x \frac{\mathrm{d}t}{t} \\ &= y^{1-s}\sigma_{2s-1}(n) \int_0^\infty \frac{\sqrt{\pi}}{\Gamma(s)} t^{s-1/2} e^{-t} e^{-(ny)^2\pi/t} \frac{\mathrm{d}t}{t} \\ &= \frac{y^{1-s}\sigma_{2s-1}(n)\sqrt{\pi}}{\Gamma(s)} \int_0^\infty t^{s-1/2} e^{-(t+(ny)^2\pi/t)} \frac{\mathrm{d}t}{t}. \end{aligned}$$

Change variables $t \mapsto \pi nyt$ so that

$$2\zeta(2s)C_{n,s} = \frac{\sigma_{2s-1}(n)\pi^s}{\Gamma(s)n^{s-1/2}}\sqrt{y} \int_0^\infty t^{s-1/2} e^{-(t+1/t)\pi ny} \frac{\mathrm{d}t}{t}.$$

Last, divide both sides by $2\zeta(2s)$, and recognize a factor of $1/\xi(2s)$ to find:

The nth Fourier component of E_s is

$$C_{n,s}(y) = \frac{\sigma_{2s-1}(n)}{n^{s-1/2}\xi(2s)}\sqrt{y} \int_0^\infty t^{s-1/2} e^{-(t+1/t)\pi ny} \frac{\mathrm{d}t}{t}.$$

Isolate the dependence of y in $C_{1,s}(y)$ and define

$$K_s(y) = \sqrt{y}\int_0^\infty t^{s-1/2} e^{-(t+1/t)\pi y}\,\frac{\mathrm{d}t}{t}.$$

Appending the constant term, we have just computed

$$E_s(x+iy) = y^s + \frac{\xi(2s-1)}{\xi(2s)}y^{1-s} + \frac{\sqrt{y}}{\xi(2s)}\sum_{n\neq 0}\frac{\sigma_{2s-1}(n)}{n^{s-1/2}}K_s(ny)e^{2\pi i n x}.$$

Eisenstein series: revisiting asymptotics

The integral expression for K_s elucidates some of the behavior of E_s. In contrast to the moderate growth of the constant term

$$c_P E_s(y) = y^s + \frac{\xi(2s-1)}{\xi(2s)}y^{1-s},$$

the Fourier components of E_s for $n \neq 0$ are of *exponential decay* as $y \to \infty$, seen as follows: when $y \gg 0$ we have $e^{-(t+1/t)\pi y} \leq e^{-(t+1/t)}e^{-\pi y}$, so we may estimate

$$|K_s(y)| \leq \sqrt{y}e^{-\pi y}\int_0^\infty t^{\mathrm{Re}(s-1/2)}e^{-(t+1/t)}\,\frac{\mathrm{d}t}{t} = \sqrt{y}e^{-\pi y}K_{\mathrm{Re}(s-1/2)}(1/\pi).$$

Thus, $K_s(y)$ is of exponential decay as $y \to \infty$. Furthermore, the function

$$y \longmapsto K_s(|n|y) = \sqrt{y}\int_0^\infty t^{s-1/2}e^{-(t+1/t)\pi|n|y}\,\frac{\mathrm{d}t}{t},$$

satisfies the differential equation characterizing the nth Fourier component,

$$u'' - (4\pi^2 n^2 + \frac{\lambda_s}{y^2})u = 0.$$

Consequently, up to the numerical coefficients computed in the last section, $y \mapsto K_s(ny)$ is the nth Fourier component of E_s. Thus for each nonzero n, the nth Fourier component is of exponential decay as claimed.

Foreshadowing cuspforms

Evidently the moderate growth of E_s, and thereby its failure to be square integrable on $\Gamma\backslash\mathfrak{H}$, is due entirely to the constant term. This phenomenon leads one to speculate that a sufficient condition on an eigenfunction f of $\Delta^{\mathfrak{H}}$ to be in $L^2(\Gamma\backslash\mathfrak{H})$ is for its constant term to vanish:

$$c_P f(y) = \int_0^1 f(x+iy)\,\mathrm{d}x = 0 \quad \text{for all } y > 0.$$

Slightly modified, the criterion in the display above characterizes *cuspforms*, though we postpone details.

Surprisingly, a stronger claim can be made for cuspform eigenfunctions: an eigenfunction f of $\Delta^{\mathfrak{H}}$ such that $c_P f(y) = 0$ for all $y > 0$ is of *exponential decrease* as $y \to \infty$, proved as follows.

Take a $\Delta^{\mathfrak{H}}$ function $f \in C^\infty(\Gamma\backslash\mathfrak{H})$ with eigenvalue λ, and suppose $c_P f(y) = 0$ for all $y \geq 0$. Further, suppose f is of moderate growth, in the sense that there are constants A and C such that $|f(x+iy) \leq Cy^A$ as $y \to \infty$.

The nth (for $n \neq 0$) Fourier component of f satisfies the differential equation obtained by equating components of $\Delta^{\mathfrak{H}} f = \lambda f$. Up to a constant multiple $c_{n,\lambda}$ this is the same differential equation characterizing the Fourier components of E_s. Since f is of *moderate growth*, the nth Fourier component of f is

$$y \mapsto c_{n,\lambda} K_s(|n|y) = c_{n,\lambda}\sqrt{y}\int_0^\infty t^{s-1/2} e^{-(t+1/t)\pi|n|y}\,\frac{dt}{t}.$$

As argued, K_s is of *exponential* decay. To prove that f itself is of exponential decay, use the exponential decay of K_s to conclude the obvious

$$|c_{n,\lambda} K_s(|n|y)| = |\int_0^1 e^{-2\pi inx} f(x+iy)| \leq Cy^A \qquad \text{as } y \to \infty.$$

Then, for y_0 big enough, using the exponential decay of K_s (and thereby exponential growth of $1/K_s$), altering C as needed, obtain

$$|c_{n,\lambda}| \leq C\frac{y_0^A}{K_s(|n|y_0)} \leq Cy_0^A e^{2\pi|n|y_0}.$$

Replacing C by Cy_0^A, recognizing a geometric series, we have

$$|f(x+iy)| \leq C\sum_{n\neq 0} y_0 e^{2\pi|n|y_0} e^{-2\pi|n|y} = C\frac{2e^{-2\pi(y-y_0)}}{1-e^{-2\pi(y-y_0)}}.$$

For $y \gg y_0$, this is

$$|f(x+iy)| \leq Ce^{-2\pi y},$$

showing that f is of *exponential decay.*

Cuspform eigenfunctions of $\Delta^{\mathfrak{H}}$ are of *rapid decay* and thus in $L^2(\Gamma\backslash\mathfrak{H})$. This is in contrast to the not-square integrable eigenfunctions E_s which grow moderately. In the next two sections, we show that these two phenomena completely describe $L^2(\Gamma\backslash\mathfrak{H})$: cuspform eigenfunctions form an orthonormal basis of the space of cuspforms $L^2_{\text{cfm}}(\Gamma\backslash\mathfrak{H})$, the latter being the orthogonal complement of $L^2_{\text{cts}}(\Gamma\backslash\mathfrak{H})$, a space spanned by suitable integrals against Eisenstein series.

4.1.2 Continuous spectrum: pseudo-Eisenstein series

The Eisenstein series E_s has real, non-positive eigenvalue only when $\mathrm{Re}(s) = 1/2$ or $s \in [0,1]$. Our computation of the constant term showed that the growth of E_s as $y \to \infty$ is slowest precisely for $\mathrm{Re}(s) = 1/2$. This is analogous to the oscillations $x \mapsto e^{i2\pi x\xi}$ on $\mathbb{R}$ having real, non-positive eigenvalues only when $\xi \in \mathbb{R}$, being the only ξ for which that function is *bounded*.

The collection $\{E_{1/2+it}\}_{t\in\mathbb{R}}$ does play an analogous role to $\{e^{2\pi ix\xi}\}_{\xi\in\mathbb{R}}$ in the spectral decomposition of $L^2(\Gamma\backslash\mathfrak{H})$, in the sense that the continuous part of the spectrum of $\Delta^{\mathfrak{H}}$ in $L^2(\Gamma\backslash\mathfrak{H})$ is spanned by integrals of functions against $E_{1/2+it}$, through a process akin to the Fourier transform.

pseduo-Eisenstein series through adjunction

In taking the constant term of the Eisenstein series, there was no issue with thinking of $c_P E_s$ as a genunie *function* of y, because E_s is *smooth*. Rather than resorting an *almost-everywhere* description of the constant term, we think of $c_P(\cdot)$ as a map between distributions, taking functionals

$$u_f : \Phi \longmapsto \int_{\Gamma\backslash\mathfrak{H}} \Phi(x+iy) f(x+iy) \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}$$

on $C_c^\infty(\Gamma\backslash\mathfrak{H})$ to functionals

$$c_P(u_f) : \varphi \mapsto \int_0^\infty \varphi(y) c_P f(y) \frac{\mathrm{d}y}{y^2}$$

on $C_c^\infty(0,\infty)$. Recalling that $N = \{\left[\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right] : x \in \mathbb{R}\}$, the measure $\frac{\mathrm{d}y}{y^2}$ arises from thinking of $(0,\infty)$ as the quotient $N \setminus \mathfrak{H}$, and retaining the measure from the latter.

To reduce clutter, let $\langle\cdot,\cdot\rangle_1$ denote the bilinear pairing of distributions with functions on $\Gamma\backslash\mathfrak{H}$ and $\langle\cdot,\cdot\rangle_2$ denote the pairing on $(0,\infty)$.

Anticipating a characterization of cuspforms as $f \in L^2(\Gamma\backslash\mathfrak{H})$ such that $c_P f = 0$ distributionally, it is convenient to describe their complement through an adjunction. That is, we look for a map $\Psi_\cdot : C_c^\infty(0,\infty) \to C_c^\infty(\Gamma\backslash\mathfrak{H})$ satisfying

$$\langle c_P f, \varphi\rangle_2 = \langle f, \Psi_\varphi\rangle_1 \qquad \text{for all } f \in L^2(\Gamma\backslash\mathfrak{H}) \text{ and } \varphi \in C_c^\infty(0,\infty).$$

Supposing it exists, the smooth function Ψ_φ is called the **pseudo-Eisenstein series** with datum φ.

We can construct $\Psi_\cdot$ directly, proving its existence, as follows: take $\varphi \in C_c^\infty(0,\infty)$ and $f \in L^2(\Gamma\backslash\mathfrak{H})$, then the pairing is

$$\langle c_P f, \varphi\rangle_2 = \int_0^\infty c_P f(y)\varphi(y) \frac{\mathrm{d}y}{y^2}.$$

Thinking of $c_P f$ as coming from the *average* $\int_0^1 f(x+iy)\,\mathrm{d}x$ of the Γ-invariant f, the right side of the display unwinds to an integral over $\Gamma \cap P \setminus \mathfrak{H}$,

$$\langle c_P f, \varphi\rangle_2 = \int_{\Gamma\cap P\backslash\mathfrak{H}} f(x+iy)\varphi(\mathrm{Im}(x+iy))\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.$$

Winding up to to an integral of a $\Gamma \cap P$ invariant function on $\Gamma\backslash\mathfrak{H}$,

$$\langle c_P f, \varphi\rangle_2 = \int_{\Gamma\backslash\mathfrak{H}} \sum_{\gamma\in P\cap\Gamma\backslash\Gamma} f(\gamma(x+iy))\varphi(\mathrm{Im}(\gamma(x+iy))\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.$$

Since f is Γ-invariant, it passes through the sum, so

$$\langle c_P f, \varphi\rangle_2 = \int_{\Gamma\backslash\mathfrak{H}} f(x+iy) \sum_{\gamma\in P\cap\Gamma\backslash\Gamma} \varphi(\mathrm{Im}(\gamma(x+iy))\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.$$

Granting convergence[8] , define the Γ-invariant function

$$\Psi_\varphi(x+iy) = \sum_{\gamma\in P\cap\Gamma\backslash\Gamma} \varphi(\mathrm{Im}(\gamma(x+iy))).$$

We have found

$$\langle c_P f, \varphi\rangle_2 = \langle f, \Psi_\varphi\rangle_1,$$

showing $\Psi.$ fits into the adjunction as desired.

Decomposition of Ψ_φ: Mellin inversion of datum

The Mellin transform[9] is

$$\mathcal{M}\varphi : i\xi \longmapsto \mathcal{M}\varphi(i\xi) = \int_0^\infty \varphi(r)r^{-i\xi}\frac{\mathrm{d}r}{r} \qquad \text{for } \varphi \in C_c^\infty(0,\infty).$$

For smooth, compactly supported φ, we can extend $\mathcal{M}\varphi$ to a function on $\mathbb{C}$ rather than just the imaginary line. For such a φ, *Mellin inversion* is a consequence of Fourier inversion,

$$\varphi(y) = \frac{1}{2\pi i}\int_{-\infty}^\infty \mathcal{M}\varphi(i\xi)y^{i\xi}\,\mathrm{d}\xi.$$

Viewing ξ as the imaginary part of the complex variable $s = 0 + i\xi$, we can rewrite Mellin inversion as a *path integral*

$$\varphi(y) = \frac{1}{2\pi i}\int_{0-i\infty}^{0+i\infty} \mathcal{M}\varphi(i\xi)y^s\,\mathrm{d}s.$$

[8]Convergence follows from the sum being *locally finite*: for $x+iy$ in a fixed compact $\mathfrak{H}$, only finitely many summands in Ψ_φ are nonzero.

[9]This Mellin transform is the Fourier transform in *multiplicative coordinates*. For a suitable function f on $\mathbb{R}$, take $f(x) = \varphi(e^x)$ for φ on $(0,\infty)$. Then $\mathcal{F}f(\xi) = \mathcal{M}\varphi(i\xi)$

As discussed in [14], for smooth compactly supported φ, the path can be moved without disrupting Mellin inversion

$$\varphi(y) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(i\xi)y^s\,\mathrm{d}s \qquad \text{for all } \sigma\in\mathbb{R}.$$

Granting this, the pseudo-Eisenstein series with datum φ decomposes: for any $\sigma\in\mathbb{R}$, we have

$$\Psi_\varphi(x+iy) = \frac{1}{2\pi i}\sum_{\gamma\in\Gamma\cap P\backslash\Gamma}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s)\cdot(\operatorname{Im}\gamma(x+iy))^s\,\mathrm{d}s.$$

Recognizing the dependence of the integrand on γ as the summands in Eisenstein series, ake $\sigma = \operatorname{Re}(s) > 1$, so that integral is absolutely convergent, then exchange the integral and sum

$$\begin{aligned}\Psi_\varphi(x+iy) &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s)\sum_{\gamma\in\Gamma\cap P\backslash\Gamma}(\operatorname{Im}\gamma(x+iy))^s\,\mathrm{d}s\\ &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s)E_s\,\mathrm{d}s.\end{aligned}$$

Thus, the pseudo-Eisenstein series attached to φ is expressible as an integral against E_s. Since the datum φ is auxiliary, we prefer an expression of expressing the pseudo-Eisentein series as an integral of Ψ_φ against E_s.

Decomposition of Ψ_φ: characterization of E_s

The adjunction characterizing the pseudo-Eisenstein series attached to $\varphi\in C_c^\infty(0,\infty)$ is

$$\langle c_P f, \varphi\rangle_2 = \langle f, \Psi_\varphi\rangle_1.$$

From the explicit formula for Ψ_φ, a natural heuristic is that E_s is the 'pseudo-Eisentein series attached to the function $y\mapsto y^s$,' so that it fits into the adjunction

$$\langle c_P f, y^s\rangle_2 = \langle f, E_s\rangle_1.$$

However, this is not immediately sensible: the function $y\mapsto y^s$ is not compactly supported. Nonetheless, the adjunction does hold. For $\operatorname{Re}(s)>1$, it is proven by the same technique we used to construct the pseudo-Eisenstein series. Then one may use the analytic continuation of E_s to obtain the adjunction for all $s\in\mathbb{C}$.

Granting the adjunction, we see for $f\in L^2(\Gamma\backslash\mathfrak{H})$ and all $s\in\mathbb{C}$, that

$$\langle f, E_s\rangle_1 = \langle c_P f, y^s\rangle_2 = \int_0^\infty c_P f(iy)y^s\frac{\mathrm{d}y}{y^2} = \int_0^\infty c_P f(iy)y^{-(1-s)}\frac{\mathrm{d}y}{y}.$$

Recognizing the latter expression as the Mellin transform of $c_P f$ at $1-s$, we have just found

$$\langle f, E_s\rangle_1 = \mathcal{M}c_P f(1-s).$$

In particular, letting $f=\Psi_\varphi$, this is

$$\langle \Psi_\varphi, E_s\rangle_1 = \mathcal{M}(c_P\Psi_\varphi)(1-s).$$

Decomposition of Ψ_φ: moving σ to $1/2$, accounting for residues

Thinking of the path integral in the expression

$$\Psi_\varphi(x+iy) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \mathcal{M}\varphi(s)E_s\,\mathrm{d}s \qquad \sigma>1$$

as the limit of path integrals around tall rectangles facing right, moving σ to the *left* past $\sigma=1$ to $\sigma=1/2$ will pick up any resides of $E_s\mathcal{M}\varphi(s)$ in the half plane $\sigma>1/2$. That is,

$$\Psi_\varphi(x+iy) = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \mathcal{M}\varphi(s)E_s\,\mathrm{d}s + \text{ residues }.$$

For any smooth compactly supported φ, the function $\mathcal{M}\varphi$ is *entire*, so the only poles of $E_s\mathcal{M}\varphi(s)$ in the half plane $\sigma>1/2$ are at the poles of E_s. As computed in the section on analytic continuation of E_s, the only pole in that region is at $s=1$. The residue $\operatorname{res}_{s=1} E_s$ is the constant function $\pi/3$. Last, compute

$$\mathcal{M}\varphi(1) = \int_0^\infty \varphi(y)y^{-1}\frac{\mathrm{d}y}{y} = \langle 1,\varphi\rangle_2.$$

Viewing 1 as the constant term of the function $x+iy\mapsto 1$, the adjunction says

$$\mathcal{M}\varphi(1) = \langle 1,\varphi\rangle_2 = \langle 1,\Psi_\varphi\rangle_1.$$

Consequently,

$$\Psi_\varphi(x+iy) = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \mathcal{M}\varphi(s)E_s\,\mathrm{d}s + \langle 1,\Psi_\varphi\rangle_1\pi/3.$$

Decomposition of Ψ_φ: rewriting $\mathcal{M}\varphi$ in terms of Ψ_φ

We want to rewrite

$$\Psi_\varphi(x+iy) = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \mathcal{M}\varphi(s)E_s\,\mathrm{d}s + \langle 1,\Psi_\varphi\rangle_1\pi/3$$

in terms of Ψ_φ rather than φ itself.

First, the characterization of E_s yields

$$\langle E_s,\Psi_\varphi\rangle_1 = \mathcal{M}c_P\Psi_\varphi(1-s).$$

But also, the characterization of Ψ_φ shows

$$\langle E_s,\Psi_\varphi\rangle_1 = \langle c_PE_s,\varphi\rangle_2.$$

Writing $c_s=\xi(2s-1)/\xi(2s)$, the constant term of E_s is $y^s+c_sy^{1-s}$. Continue,

$$\langle E_s,\Psi_\varphi\rangle_1 = \langle y^s+c_sy^{1-s},\varphi\rangle_2 = \langle y^s,\varphi\rangle_2 + c_sy^{1-s}\varphi_2.$$

Expanding the bilinear forms and recalling the definition of the Mellin transform, this is

$$\langle E_s, \Psi_\varphi\rangle_1 = \mathcal{M}\varphi(1-s) + c_s\mathcal{M}\varphi(s).$$

Pairing these two computations replacing s by $1-s$, we have

$$\mathcal{M}(c_P\Psi_\varphi)(s) = \mathcal{M}\varphi(s) + c_{1-s}\mathcal{M}\varphi(1-s).$$

To apply the formula in this previous display, break up the integral in

$$\Psi_\varphi(x+iy) = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \mathcal{M}\varphi(s)E_s\,\mathrm{d}s + \langle 1, \Psi_\varphi\rangle_1\pi/3$$

into integrals from $1/2-i\infty$ to $1/2+i0$ and $1/2+i0$ to $1/2+i\infty$. In the former, replace s by $1-s$ to transform it to integral from $1/2+i0$ to $1/2+i\infty$. Then

$$\Psi_\varphi(x+iy) = \frac{1}{2\pi i}\int_{1/2+i0}^{1/2+i\infty} (\mathcal{M}\varphi(1-s)E_{1-s} + \mathcal{M}\varphi(s)E_s)\,\mathrm{d}s + \langle 1, \Psi_\varphi\rangle_1\pi/3.$$

Recalling $c_s = \xi(2s-1)/\xi(2s$, the functional equation for E_s rewrites to $E_{1-s} = c_{1-s}E_s$. Applying this, compute

$$\Psi_\varphi(x+iy) = \frac{1}{2\pi i}\int_{1/2+i0}^{1/2+i\infty} (c_{1-s}\mathcal{M}\varphi(1-s) + \mathcal{M}\varphi(s))E_s\,\mathrm{d}s + \langle 1, \Psi_\varphi\rangle_1\pi/3.$$

Recognize the factor in the integrand as the previously computed $\mathcal{M}c_P\Psi_\varphi(s) = \langle E_{1-s}, \Psi_\varphi\rangle_1$ we have

$$\begin{aligned}\Psi_\varphi(x+iy) &= \frac{1}{2\pi i}\int_{1/2+i0}^{1/2+i\infty} \mathcal{M}c_P\Psi_\varphi(s)E_s\,\mathrm{d}s + \langle 1, \Psi_\varphi\rangle_1\pi/3\\ &= \frac{1}{2\pi i}\int_{1/2+i0}^{1/2+i\infty} \langle E_{1-s}, \Psi_\varphi\rangle_1 E_s\,\mathrm{d}s + \langle 1, \Psi_\varphi\rangle_1\pi/3.\end{aligned}$$

As a final modification, we make the pairing $\langle E_{1-s}, \cdot\rangle_1$ *herimitian*, by noting that $\overline{E_{1-s}} = E_s$ (using the functional equation), and reverting the integral to the whole line $\mathrm{Re}(s) = 1/2$ (introducing a factor of $1/2$). All together, this is:

For any $\varphi \in C_c^\infty(0,\infty)$,

$$\Psi_\varphi = \frac{1}{4\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \langle \Psi_\varphi, E_s\rangle_1 E_s\,\mathrm{d}s + \langle 1, \Psi_\varphi\rangle_1\pi/3.$$

Letting $L^2_{\text{cts}}(\Gamma\backslash\mathfrak{H})$ denote the completion in $L^2(\Gamma\backslash\mathfrak{H})$ of finite linear combinations of pseudo-Eisenstein series, we have just shown that

$$L^2_{\text{cts}}(\Gamma\backslash\mathfrak{H}) \approx L^2(1/2+i\mathbb{R}) \oplus \mathbb{C}.$$

The isomorphism is the isometric extension of

$$f \longmapsto \langle f, E_s\rangle_1 \oplus \frac{\pi}{3}\langle f, 1\rangle_1 \quad \text{for } f \in L^2_{\text{cts}}(\Gamma\backslash\mathfrak{H}).$$

4.1.3 Cuspforms

In computing the Fourier series for E_s, we found that its moderate growth (and thereby failure to be square integrable) is dictated by its constant term $c_P E_s(y) = y^s + c_s y^{1-s}$. With the exception of the constant term, the rest of the Fourier components are rapidly decaying, and we found that this was true for *all* $\Delta^{\mathfrak{H}}$ eigenfunctions on $\Gamma\backslash\mathfrak{H}$. Consequently, barring a possible conflict with numerical coefficients, we anticipate that $\Delta^{\mathfrak{H}}$ eigenfunctions with 'vanishing' constant term will be of rapid decay.

A **cuspform** $f \in L^2(\Gamma\backslash\mathfrak{H})$ is characterized by its constant term vanishing *as a distribution*

$$\langle c_P f, \varphi\rangle_2 = 0 \qquad \text{for all } \varphi \in C_c^\infty(0,\infty).$$

From the adjunction characterizing the pseudo-Eisenstein series, we see immediately

$$f \text{ is a cuspform} \iff 0 = \langle c_P f, \varphi\rangle_2 = \langle f, \Psi_\varphi\rangle_1 \qquad \text{for all } \varphi \in C_c^\infty(0,\infty).$$

That is, the space of cuspforms $L^2_{\text{cfm}}(\Gamma\backslash\mathfrak{H})$ is the orthogonal complement of pseudo-Eisenstein series. Recalling that the $L^2(\Gamma\backslash\mathfrak{H})$ completion of finite linear combinations of pseudo-Eisenstein series is denoted $L^2_{\text{cts}}(\Gamma\backslash\mathfrak{H})$, we have just shown

$$L^2(\Gamma\backslash\mathfrak{H}) = L^2_{\text{cfm}}(\Gamma\backslash\mathfrak{H}) \oplus L^2_{\text{cts}}(\Gamma\backslash\mathfrak{H}).$$

In this section, we show that the space of cuspforms has an orthonormal basis of eigenfunctions of $\Delta^{\mathfrak{H}}$, determining a complete spectral decomposition of $L^2(\Gamma\backslash\mathfrak{H})$.

Approaching cuspforms: $c_P f$ vanishing above a fixed hight

Rather than addressing $L^2_{\text{cfm}}(\Gamma\backslash\mathfrak{H})$ directly, we analyze spaces of functions with constant terms supported away from the interval $[a,\infty)$. To this end, introduce a subspace of the pseudo-Eisenstein series

$$\Psi_{\geq a} = \{\Psi_\varphi : \varphi \in C_c^\infty(0,\infty),\ \varphi \text{ supported inside } [a,\infty).$$

By the adjunction of distributions

$$\langle c_P f, \varphi\rangle_2 = \langle f, \Psi_\varphi\rangle_1,$$

the orthogonal complement of $\Psi_{\geq a}$ consists of $f \in L^2(\Gamma\backslash\mathfrak{H})$ such that the support of the distribution $c_P f$ is contained in $(0,a)$. From this we define the space of almost-cuspforms, with 'constant term vanishing above a' as

$$L^2_a(\Gamma\backslash\mathfrak{H}) = \text{ orthogonal complement to } \Psi_{\geq a}.$$

In [9] it is proven that $C_a^\infty(\Gamma\backslash\mathfrak{H}) = C_c^\infty(\Gamma\backslash\mathfrak{H}) \cap L^2_a(\Gamma\backslash\mathfrak{H})$ is dense in $L^2_a(\Gamma\backslash\mathfrak{H})$. Restrict $\Delta^{\mathfrak{H}}$ to an operator $\Delta^{\mathfrak{H}}_a$ on $C_a^\infty(\Gamma\backslash\mathfrak{H})$.

Compactness of the resolvent (of the extension (of the restriction))

The symmetry and negative semi-definiteness of $\Delta^{\mathfrak{H}}$ endow $\Delta^{\mathfrak{H}}_a$ with the same properties. By Friedrichs' construction $\Delta^{\mathfrak{H}}_a$ has a self-adjoint extension $\widetilde{\Delta^{\mathfrak{H}}}_a$, defined on a dense subspace of $H^1_a(\Gamma\backslash\mathfrak{H})$, the completion of $C^\infty_a(\Gamma\backslash\mathfrak{H})$ with respect to the norm

$$|f|^2_{H^1_a(\Gamma\backslash\mathfrak{H})} = \langle (1-\Delta^{\mathfrak{H}}_a)f, f\rangle.$$

The extension $\widetilde{\Delta^{\mathfrak{H}}}_a$ is characterized by the relation with its resolvent

$$\langle f, g\rangle = \langle (\lambda - \widetilde{\Delta^{\mathfrak{H}}}_a)^{-1} f, (1-\Delta^{\mathfrak{H}}_a)g\rangle \quad \text{with } f \in L^2_a(\Gamma\backslash\mathfrak{H}) \text{ and } g \in C^\infty_a(\Gamma\backslash\mathfrak{H}).$$

By construction, the resolvent $(\lambda - \widetilde{\Delta^{\mathfrak{H}}}_a)^{-1}$ is a continuous map $L^2_a(\Gamma\backslash\mathfrak{H}) \to H^1_a(\Gamma\backslash\mathfrak{H})$. To prove that the resolvent is compact, it suffices to prove that $H^1_a(\Gamma\backslash\mathfrak{H}) \to L^2_a(\Gamma\backslash\mathfrak{H})$ is compact.

Claim 11. For $a > 0$, the inclusion $H^1_a(\Gamma\backslash\mathfrak{H}) \to L^2_a(\Gamma\backslash\mathfrak{H})$ is compact.

Proof. To prove that $H^1_a(\Gamma\backslash\mathfrak{H}) \to L^2_a(\Gamma\backslash\mathfrak{H})$ is compact, we show that the unit ball in $H^1_a(\Gamma\backslash\mathfrak{H})$ is totally bounded in $L^2_a(\Gamma\backslash\mathfrak{H})$. In order to make reference to pointwise values, we prove the corresponding claim on smooth functions: $B = \{f \in C^\infty_a(\Gamma\backslash\mathfrak{H}) : |f|_{H^1_a(\Gamma\backslash\mathfrak{H})} \le 1\}$ is totally bounded in $L^2_a(\Gamma\backslash\mathfrak{H})$.

Fix $\varepsilon \ge 0$ and take c such that $1/c^2 < \varepsilon$. Let Y_0 be the image of $\sqrt{3}/2 \le y \le c+1$ and Y_∞ be the image of $y \ge c$ under the projection $\mathfrak{H} \mapsto \Gamma\backslash\mathfrak{H}$. Cover $\Gamma\backslash\mathfrak{H}$ with small coordinate patches U_i. Since Y_0 is compact, reduce to a finite cover $U_1, ..., U_n$ (of Y_0). Combine the complementary opens to obtain a cover U_∞ of Y_∞.

Take a smooth partition of $\varphi_1, ..., \varphi_n, \varphi_\infty$ unity subordinate to the cover $U_1, ..., U_n, U_\infty$, with φ_∞ identically 1 for $y \ge c$. Then $B = \varphi_1 \cdot B + ... + \varphi_n \cdot B + \varphi_\infty \cdot B$. For $i \ne \infty$, the functions $\varphi_i \cdot B$ may be identified with functions on $\mathbb{T}^2$. By similar local arguments to those on the sphere, $\varphi_i \cdot B$ is bounded in $H^1(\mathbb{T}^2)$, and is thus totally bounded in $L^2(\mathbb{T}^2)$, by classical Rellich compactness. Include $L^2(\mathbb{T}^2)$ in $L^2_a(\Gamma\backslash\mathfrak{H})$ to show that $\varphi_i \cdot B$ is covered by finitely many ε-balls in $L^2_a(\Gamma\backslash\mathfrak{H})$. Repeating for each $i \ne \infty$ shows $(1-\varphi_\infty)B$ is covered by finitely many ε-balls in $L^2_a(\Gamma\backslash\mathfrak{H})$.

To prove B is totally bounded in $L^2_a(\Gamma\backslash\mathfrak{H})$, we show that $\varphi_\infty \cdot B$ is contained in a single ε ball in $L^2_a(\Gamma\backslash\mathfrak{H})$. That is, we seek to show

$$\int_{y>c} |\varphi_\infty \cdot f|^2 \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} < \varepsilon \quad \text{for all } f \in B.$$

Elementary inequalities show that the $H^1_a(\Gamma\backslash\mathfrak{H})$ norm of $\varphi_\infty \cdot f$ is bounded by the $H^1_a(\Gamma\backslash\mathfrak{H})$ norm of f itself, so it suffices to prove the inequality in the display above for any $f \in B$.

Let $C_{n,f}(y)$ denote the nth Fourier component of f, a function of y. The Fourier components are designed to satisfy $2\pi i n C_{n,f}(y) = C_{n,f_x}(y)$. The Plancherel identity says

$\int_0^1 |f(x+iy)|^2\,\mathrm{d}x = \sum_{n\in\mathbb{Z}} |C_{n,f}(y)|^2$. Further, since $c \geq a$, the constant term $C_{0,f}(y)$ vanishes identically when $y \geq c$. Compute directly,

$$\int\int_{y>c} |f|^2 \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \leq \frac{1}{c^2}\int\int_{y>c} |f|^2\,\mathrm{d}x\,\mathrm{d}y.$$

Using Plancherel, noting that $C_{0,f}(y)$ is zero on the region of integration, and then using the noted relationship between differentiation and multiplication-by-n on the spectral side,

$$\begin{aligned}\int\int_{y>c} |f|^2 \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} &\leq \frac{1}{c^2}\sum_{n\neq 0}\int_{y>c} |C_{n,f}(y)|^2\,\mathrm{d}y \leq \frac{1}{c^2}\sum_{n\neq 0}(2\pi n)^2\int_{y>c}|C_{n,f}(y)|^2\,\mathrm{d}y \\ &\leq \frac{1}{c^2}\sum_{n\neq 0}\int_{y>c} |C_{n,f_x}(y)|^2\,\mathrm{d}y.\end{aligned}$$

Using Plancherel in the other direction, then integrating by parts, appending the positive quantity $\int\int |f_y|^2\,\mathrm{d}x\,\mathrm{d}y$, we have

$$\begin{aligned}\int\int_{y>c} |f|^2 \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} &\leq \frac{1}{c^2}\int\int_{y>c} \left|\frac{\partial f}{\partial x}\right|^2\mathrm{d}x\,\mathrm{d}y = \frac{1}{c^2}\int\int_{y>c} -\frac{\partial^2 f}{\partial x^2}\cdot\overline{f}\,\mathrm{d}x\,\mathrm{d}y \\ &\leq \frac{1}{c^2}\int\int_{y>c} -\frac{\partial^2 f}{\partial x^2}\cdot\overline{f} - \frac{\partial^2 f}{\partial y^2}\cdot\overline{f}\,\mathrm{d}x\,\mathrm{d}y = -\int\int_{y>c}\Delta^{\mathfrak{H}} f\cdot\overline{f}\frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.\end{aligned}$$

Next, integrating over all of $\Gamma\backslash\mathfrak{H}$, we see

$$\int\int_{y>c} |f|^2 \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \leq \frac{1}{c^2}\int\int_{\Gamma\backslash\mathfrak{H}} -\Delta^{\mathfrak{H}} f\cdot\overline{f}\frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \leq \frac{1}{c^2}|f|^2_{H^1_a(\Gamma\backslash\mathfrak{H})}.$$

Last, since $f \in B$, the smooth functions in the unit ball of $H^1_a(\Gamma\backslash\mathfrak{H})$, we have just shown that

$$\int\int_{y>c} |f|^2 \frac{\mathrm{d}x\,\mathrm{d}y}{y^2} \leq \frac{1}{c^2} < \varepsilon,$$

as claimed. Consequently, functions supported on $y > c$ in B are all contained in a single ε ball in $L^2_a(\Gamma\backslash\mathfrak{H})$. This completes the proof that B is totally bounded in $L^2_a(\Gamma\backslash\mathfrak{H})$, showing that the inclusion $H^1_a(\Gamma\backslash\mathfrak{H}) \to L^2_a(\Gamma\backslash\mathfrak{H})$ is *compact.* □

Consequently $L^2_a(\Gamma\backslash\mathfrak{H})$ has an orthonormal basis of $\widetilde{\Delta^{\mathfrak{H}}}_a$ eigenvectors, for all $a > 0$.

Remark 10. That $(\lambda - \widetilde{\Delta^{\mathfrak{H}}})_a^{-1}$ has only discrete spectrum indicates that suitable pseudo-Eisenstein series become genuine eigenfunctions of $\widetilde{\Delta^{\mathfrak{H}}}_a$, in striking contrast to the presence of continuous spectra for $\Delta^{\mathfrak{H}}$. This is explained by the Friedrichs' extension $\widetilde{\Delta^{\mathfrak{H}}}_a$ ability to ignore zeroth order distributions supported at $y = a$. The new eigenfunctions that leak into the discrete spectrum are truncated pseudo-Eisenstein series, which are not genuine eigenfunctions of $\Delta^{\mathfrak{H}}$ due to a slight roughness at the truncation.

Discrete decomposition of cuspforms

The subspace $L^2_a(\Gamma\backslash\mathfrak{H})$ consists of elements with constant terms supported on $(0,a)$. The intersection over a is

$$\bigcap_{a>0} L^2_a(\Gamma\backslash\mathfrak{H}) = L^2_{\text{cfm}}(\Gamma\backslash\mathfrak{H}).$$

We know that each $L^2_a(\Gamma\backslash\mathfrak{H})$ has an orthonormal basis of $\widetilde{\Delta^{\mathfrak{H}}}_a$ eigenfunctions, but as discussed in [9], each space $L^2_a(\Gamma\backslash\mathfrak{H})$ is *not* stable under $(\lambda-\widetilde{\Delta^{\mathfrak{H}}})^{-1}$. Consequently, some of the eigenfunctions of $\widetilde{\Delta^{\mathfrak{H}}}_a$ in $L^2_a(\Gamma\backslash\mathfrak{H})$ are not genuine eigenfunctions of $(\lambda-\widetilde{\Delta^{\mathfrak{H}}})^{-1}$, nor $\widetilde{\Delta^{\mathfrak{H}}}$ itself. In contrast, we show next that the common intersection $L^2_{\text{cfm}}(\Gamma\backslash\mathfrak{H})$ *is* stable under $(\lambda-\widetilde{\Delta^{\mathfrak{H}}})^{-1}$, thereby showing that $L^2_{\text{cfm}}(\Gamma\backslash\mathfrak{H})$ has an orthonormal basis of eigenfunctions of $\widetilde{\Delta^{\mathfrak{H}}}$ eigenfunctions.

Claim 12. There is some positive real λ such that $L^2_{\text{cfm}}(\Gamma\backslash\mathfrak{H})$ is stable under $(\lambda-\widetilde{\Delta^{\mathfrak{H}}})^{-1}$.

Proof. We characterized $L^2_{\text{cfm}}(\Gamma\backslash\mathfrak{H})$ as the orthogonal complement in $L^2(\Gamma\backslash\mathfrak{H})$ to the space of pseudo-Eisenstein series $L^2_{\text{cts}}(\Gamma\backslash\mathfrak{H})$. Thus for $f\in L^2_{\text{cfm}}(\Gamma\backslash\mathfrak{H})$, to show $(\lambda-\widetilde{\Delta^{\mathfrak{H}}})^{-1}f\in L^2_{\text{cfm}}(\Gamma\backslash\mathfrak{H})$, we must show that

$$\langle(\lambda-\widetilde{\Delta^{\mathfrak{H}}})^{-1}f,\Psi_\varphi\rangle = 0 \quad \text{for all } \varphi\in C_c^\infty(\Gamma\backslash\mathfrak{H}).$$

Recall that Ψ_φ is the smooth, locally finite sum

$$\Psi_\varphi(\tau) = \sum_{\gamma\in P\cap\Gamma\backslash\Gamma}\varphi(\mathrm{Im}(\gamma\tau)).$$

Consequently, $(\lambda-\widetilde{\Delta^{\mathfrak{H}}})^{-1}\Psi_\varphi = \Psi_{(\lambda-\widetilde{\Delta^{\mathfrak{H}}})^{-1}\varphi}$. But since φ is in the original domain of $\Delta^{\mathfrak{H}}$, we see $(\lambda-\widetilde{\Delta^{\mathfrak{H}}})^{-1}\varphi = (\lambda-\Delta^{\mathfrak{H}})^{-1}\varphi$.

Then, using that $\widetilde{\Delta^{\mathfrak{H}}}$ is *self-adjoint*, and that λ is *real*, we see that condition above is equivalently

$$0 = \langle f,\Psi_{(\lambda-\Delta^{\mathfrak{H}})^{-1}\varphi}\rangle \quad \text{for all } \varphi\in C_c^\infty(\Gamma\backslash\mathfrak{H}).$$

Thus, to show that $L^2_{\text{cfm}}(\Gamma\backslash\mathfrak{H})$ is stabilized under $(\lambda-\widetilde{\Delta^{\mathfrak{H}}})^{-1}$, it suffices to show that there is a class of data Φ (suitable to define nicely convergent pseudo-Eisenstein series) closed under the differential equation

$$(\lambda-\Delta^{\mathfrak{H}})\psi = \varphi \quad \text{for all } \varphi\in C_c^\infty(\Gamma\backslash\mathfrak{H}), \text{ there is a } \psi\in\Phi.$$

Since ψ and φ are functions of y alone, the differential equation is

$$\lambda\psi - y^2\frac{\partial^2}{\partial y^2}\psi = \varphi.$$

It is convenient to rewrite the equation as

$$\lambda\psi - (y\frac{\partial\psi}{\partial y})^2 + y\frac{\partial\psi}{\partial y} = \varphi.$$

Since then, parametrizing $(0,\infty)$ via additive coordinates, letting $v(t) = \varphi(e^t)$ and $u(t) = \psi(e^t)$ the equation is

$$-u'' + u' + \lambda u = v.$$

This equation transforms to

$$\xi^2\mathcal{F}u - it\mathcal{F}u + \lambda\mathcal{F}u = \mathcal{F}v,$$

or equivalently,

$$\mathcal{F}u = \frac{\mathcal{F}v}{\xi^2 - i\xi + \lambda}.$$

Since φ, and thereby v is smooth and compactly supported, Paley-Wiener asserts that $\mathcal{F}v(\xi)$ extends to an analytic function $\mathcal{F}v(\xi + i\eta)$, satisfying $|\mathcal{F}v(z)| \leq C_N(1+|z|)^{-N}e^{c|\eta|}$ for some constants c, C_N and every nonnegative integer N. Thus, for any positive M, and for $|\eta| \leq M$, the function $\xi \mapsto \mathcal{F}v(\xi + i\eta)$ is of rapid decay. The only thing barring $\mathcal{F}u$ from having identical properties is the presence of the denominator in the display above. To show that u has suitable decay properties, we show demonstrate control over M, dependent on λ (but *not* dependent on v), for which $\xi \mapsto \mathcal{F}u(\xi + i\eta)$ is of rapid decay when $|\eta| \leq M$. Granting this, Paley–Wiener shows that $t \mapsto u(t)$ itself is bounded by $e^{-M|t|}$. Changing coordinates, we would then have $\psi(y)$ is bounded by y^M as $y \to 0$ and y^{-M} as $y \to \infty$. Since we have control over M, we can[10] pick λ such that the sum

$$\Psi_\psi(x+iy) = \sum_{\gamma \in P\cap\Gamma\backslash\Gamma} \psi(\gamma(x+iy))$$

is square integrable, showing that the class of data Φ is closed under the differential equation above, and in turn that $(\lambda - \widetilde{\Delta^{\mathfrak{H}}})^{-1}f$ is indeed in the complement to $L^2_{\text{cts}}(\Gamma\backslash\mathfrak{H})$, thus a cuspform.

Letting $\lambda = s(s-1)$ be real and positive, the zeroes of $\xi^2 - i\xi + \lambda$ are at is and $i(1-s)$. Thus, for any M, we may pick a large enough real, positive s for which $(\xi + i\eta)^2 - i(\xi + i\eta) + s(s-1)$ has no roots within $|\eta| \leq M$, completing the proof as indicated in the last paragraph. □

Since $H^1_a(\Gamma\backslash\mathfrak{H}) \to L^2_a(\Gamma\backslash\mathfrak{H})$ is compact for every $a > 0$, the restriction to $H^1_{\text{cfm}}(\Gamma\backslash\mathfrak{H}) \to L^2_{\text{cfm}}(\Gamma\backslash\mathfrak{H})$ is compact. Further, by the last claim, there are λ such that $(\lambda - \widetilde{\Delta^{\mathfrak{H}}})^{-1}$ stabilizes $L^2_{\text{cfm}}(\Gamma\backslash\mathfrak{H})$. Since $(\lambda - \widetilde{\Delta^{\mathfrak{H}}})^{-1} : L^2_{\text{cfm}}(\Gamma\backslash\mathfrak{H}) \to H^1_{\text{cfm}}(\Gamma\backslash\mathfrak{H})$ is continuous, and $H^1_{\text{cfm}}(\Gamma\backslash\mathfrak{H}) \to L^2_{\text{cfm}}(\Gamma\backslash\mathfrak{H})$ is compact, we see that $(\lambda - \widetilde{\Delta^{\mathfrak{H}}})^{-1}$ is compact.

Consequently, $L^2_{\text{cfm}}(\Gamma\backslash\mathfrak{H})$ has an orthonormal basis of eigenfunctions for $(\lambda - \widetilde{\Delta^{\mathfrak{H}}})^{-1}$, thereby for $\widetilde{\Delta^{\mathfrak{H}}}$.

[10]The existence of such an M is proved in [7]

Passing from $\widetilde{\Delta^{\mathfrak{H}}}$ to $\Delta^{\mathfrak{H}}$

The author of this document has not yet found a source discussing the relevant Sobolev regularity that would permit the routine of showing that the eigenfunctions of $\widetilde{\Delta^{\mathfrak{H}}}$ must be in the original domain of $\Delta^{\mathfrak{H}}$. In fact, the author suspects that such a claim is *false*, since the discussion of asymptotics of cuspform eigenfunction Fourier components suggest that such functions cannot be compactly supported. Nonetheless, the theory of elliptic regularity for Laplace–Beltrami operators on general Riemannian manifolds *is* well developed, but a digression into differential geometry is orthogonal to the paradigm in this thesis.

Despite sacrificing the general form of the decomposition argument, the author suspects a completion of the proof that $L^2_{\text{cfm}}(\Gamma\backslash\mathfrak{H})$ decomposes into eigenspaces of $\Delta^{\mathfrak{H}}$ will follow from a careful treatment of the behavior of and relationship between the operators $\Delta^{\mathfrak{H}}$ and $(\lambda - \widetilde{\Delta^{\mathfrak{H}}}_a)^{-1}$, as follows.

Recall that $\widetilde{\Delta^{\mathfrak{H}}}_a$ is characterized by the identity

$$\langle f, g\rangle = \langle(\lambda - \Delta^{\mathfrak{H}} f, (\lambda - \widetilde{\Delta^{\mathfrak{H}}}_a)^{-1} g\rangle \quad \text{for all } g \in C_a^\infty(\Gamma\backslash\mathfrak{H}) \text{ and } f \in L_a^2(\Gamma\backslash\mathfrak{H}).$$

Working distributionally, this is

$$\langle f, g\rangle = \langle(\lambda - \Delta^{\mathfrak{H}})(\lambda - \widetilde{\Delta^{\mathfrak{H}}}_a)^{-1} f, g\rangle \quad \text{for all } g \in C_a^\infty(\Gamma\backslash\mathfrak{H}) \text{ and } f \in L_a^2(\Gamma\backslash\mathfrak{H}).$$

Subtracting,

$$0 = \langle f - (\lambda - \Delta^{\mathfrak{H}})(\lambda - \widetilde{\Delta^{\mathfrak{H}}}_a)^{-1} f, g\rangle \quad \text{for all } g \in C_a^\infty(\Gamma\backslash\mathfrak{H}) \text{ and } f \in L_a^2(\Gamma\backslash\mathfrak{H}).$$

For fixed f, this display shows that the distribution $f - (\lambda - \Delta^{\mathfrak{H}})(\lambda - \widetilde{\Delta^{\mathfrak{H}}}_a)^{-1} f$ annihilates all test functions $g \in C_a^\infty(\Gamma\backslash\mathfrak{H})$. Consequently, $f - (\lambda - \Delta^{\mathfrak{H}})(\lambda - \widetilde{\Delta^{\mathfrak{H}}}_a)^{-1} f$ has support completely contained in the *tail* Y_∞, the image of $y \geq a$ in $\Gamma\backslash\mathfrak{H}$, and consequently the non-constant-term Fourier components of $f - (\lambda - \Delta^{\mathfrak{H}})(\lambda - \widetilde{\Delta^{\mathfrak{H}}}_a)^{-1} f$ are identically zero. Moreover, since f has constant term vanishing above $y \geq a$, the distribution $f - (\lambda - \Delta^{\mathfrak{H}})(\lambda - \widetilde{\Delta^{\mathfrak{H}}}_a)^{-1} f$ does too. Thus, the distribution $f - (\lambda - \Delta^{\mathfrak{H}})(\lambda - \widetilde{\Delta^{\mathfrak{H}}}_a)^{-1} f$ has support concentrated at the point a, and since the non-constant-term Fourier components of $f - (\lambda - \Delta^{\mathfrak{H}})(\lambda - \widetilde{\Delta^{\mathfrak{H}}}_a)^{-1} f$ are identically zero (letting A denote any distribution on $(0, \infty)$ supported at a) this distribution is of the form

$$A \circ c_P : \varphi \mapsto A \circ c_P(\varphi).$$

Last, as discussed in [9], Sobolev theory requires the distribution $f - (\lambda - \Delta^{\mathfrak{H}})(\lambda - \widetilde{\Delta^{\mathfrak{H}}}_a)^{-1} f$ have order at most 1. The only distribution supported at a point with order at most 1 is the Dirac delta. Consequently, this shows that

$$f - (\lambda - \Delta^{\mathfrak{H}})(\lambda - \widetilde{\Delta^{\mathfrak{H}}}_a)^{-1} f = \delta_a \circ c_p.$$

That is, f and $(\lambda - \Delta^{\mathfrak{H}})(\lambda - \widetilde{\Delta^{\mathfrak{H}}}_a)^{-1} f$ differ by a multiple of the ‘evaluate the constant term at a’ distribution.

We view $L^2_{\text{cfm}}(\Gamma\backslash\mathfrak{H})$ as the intersection of all the $L^2_a(\Gamma\backslash\mathfrak{H})$ for $a > 0$. It seems to follow that for $f \in L^2_{\text{cfm}}(\Gamma\backslash\mathfrak{H})$, the distribution $f - (\lambda - \Delta^{\mathfrak{H}})(\lambda - \widetilde{\Delta^{\mathfrak{H}}})^{-1} f$ is the 'evaluate the constant term at 0' distribution with test function $C_0^\infty(\Gamma\backslash\mathfrak{H})$ (smooth compactly supported functions with identically zero constant term). But if f is an eigenfunction of $(\lambda - \widetilde{\Delta^{\mathfrak{H}}})^{-1}$, then this says $f - (\lambda - \Delta^{\mathfrak{H}})(1 - \lambda f) = (1 - \lambda) f - (\lambda - \Delta^{\mathfrak{H}}) f$ annihilates all of $C_0^\infty(\Gamma\backslash\mathfrak{H})$, thus is the zero distribution. Thus f is, at least distributionally, an eigenfunction of $\Delta^{\mathfrak{H}}$ itself. The transition from the space of distributions to that of genuine functions is precisely what requires a notion of Sobolev regularity.

Conclusion

Despite the lack of precise data describing cuspform eigenfunctions of $\Delta^{\mathfrak{H}}$, we have shown that the they span the orthogonal complement to pseudo-Eisenstein series.

As claimed,

$$L^2(\Gamma\backslash\mathfrak{H}) = L^2_{\text{cfm}}(\Gamma\backslash\mathfrak{H}) \oplus L^2_{\text{cts}}(\Gamma\backslash\mathfrak{H}).$$

Let $\{F_\lambda\}$ be the orthonormal basis of $\Delta^{\mathfrak{H}}$ eigenfunctions for $L^2_{\text{cfm}}(\Gamma\backslash\mathfrak{H})$, with eigenvalues λ. We have shown for $f \in L^2_a(\Gamma\backslash\mathfrak{H})$, that (with the integrals being the extension by isometry from test functions)

$$f = \sum_\lambda \langle f, F_\lambda \rangle F_\lambda + \langle f, 1 \rangle \frac{\pi}{3} + \int_{1/2 - i\infty}^{1/2 + i\infty} \langle f, E_s \rangle \, \mathrm{d}s.$$

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