SPECTRAL RIGIDITY AND FLEXIBILITY OF HYPERBOLIC MANIFOLDS

by

Justin Katz

A Dissertation

Submitted to the Faculty of Purdue University

In Partial Fulfillment of the Requirements for the degree of

Doctor of Philosophy

Department of Mathematics

West Lafayette, Indiana

August 2023

THE PURDUE UNIVERSITY GRADUATE SCHOOL
STATEMENT OF COMMITTEE APPROVAL

Dr. David Benjamin McReynolds, Co-Chair

School of Mathematics

Dr. Freydoon Shahidi, Co-Chair

School of Mathematics

Dr. Sai Kee Yeung

School of Mathematics

Dr. Lvhou Chen

School of Mathematics

Approved by:

Dr. Plamen Stefanov

TABLE OF CONTENTS

ABSTRACT

In the first part of this thesis we show that, for a given non-arithmetic closed hyperbolic n manifold M, there exist for each positive integer j, a set $M_1, ..., M_j$ of pairwise nonisometric, strongly isospectral, finite covers of M, and such that for each i, i' one has isomorphisms of cohomology groups $H^*(M_i, \mathbb{Z}) = H^*(M_{i'}, \mathbb{Z})$ which are compatible with respect to the natural maps induced by the cover. In the second part, we prove that hyperbolic 2- and 3-manifolds which arise from principal congruence subgroups of a maixmal order in a quaternion algebra having type number 1 are absolutely spectrally rigid. One consequence of this is a partial answer to an outstanding question of Alan Reid, concerning the spectral rigidity of Hurwitz surfaces.

1. INTRODUCTION

Following Milnor's construction **Milnor-[EigenvaluesLaplaceOperator]1964** of a nonisometric pair of isospectral 16-dimensional flat tori, two enterprises have emerged: to produce as large a family as possible of pairwise nonisometric, isospectral Riemannian manifolds and to prove that particular a particular closed Riemannian manifold is not isospectral to any other.

The main theorems of this thesis concern these two problems, respectively.

In the first we construct arbitrarily large families of nonarithmetic hyperbolic n-manifolds which, in addition to being isospectral, share several additional topological invariants. This construction uses an integral refinement of Gasssmann-Sunada equivalence, which was recently used by D. Prasad **Prasad-[RefinedNotionArithmetically]2017** to produce nonisomorphic numberfields which have isomorphic idele class groups.

In the second, we prove that particular families of hyperbolic surfaces and 3-manifolds are absolutely spectrally rigid. These families from principal congruence subgroups of maximal orders in quaternion algebras which satisfy a certain arithmetic condition. An application of this theorem is a partial positive answer to a question of A. Reid **Reid-[TracesLengthsAxes]2014** concerning spectral rigidity of so-called Hurwitz surfaces: those surfaces of genus g having automorphism group of order $84(g-1)$, which is the maximum possible by Hurwitz's theorem.

2. BACKGROUND

2.1 Geometry

In this section, (M, g) is a closed Riemannian manifold. We write $\mathrm{d}v_g$ for the volume form associated to g.

2.1.1 Laplace-spectrum

The eigenvalues of the Laplace operator of the Laplace operator $\Delta_{M,g}$ acting on $L^2(M,\, \mathrm{d}v_g)$ form a discrete set $\mathrm{spec}_{M,g}$ of nonnegative real numbers, tending to ∞. For each $\lambda \in \mathrm{spec}_{M,g}$, the dimension $m_{\Delta_{M,g}}(\lambda)$ of the λ-eigenspace $E_{M,g}(\lambda) = \ker(\Delta_{M,g} - \lambda\,\mathrm{id})$ in $L^2(M,\, \mathrm{d}v_g)$ is finite. We encode the this data in the **Laplace-spectral counting function**, defined for $x \in \mathbb{R}_{\geq 0}$ by

$$\pi_{\Delta_{M,g}}(x) = \sum_{\lambda \in \mathrm{spec}(\Delta_{M,g}) \cap [0,x]} \dim \ker(\Delta_{M,g} - \lambda\,\mathrm{id}) \tag{2.1}$$

2.1.2 Length spectrum

Within each free homotopy class γ of closed curves on M there is a geodesic representative of minimal length; we write $\ell_g(\gamma)$ for that length. For a positive real number l, we write $\mathrm{Lspec}\, M, g(\ell)$ for the number of free homotopy classes γ of closed curves in M for which $\ell_g(\gamma) = l$.

Two compact Riemannian manifolds (M, g), (M', g') are said to be **Laplace-isospectral** (resp. **length-isospectral**) if $\mathrm{spec}_{M,g} = \mathrm{spec}_{M',g'}$ (resp. if $\mathrm{Lspec}_{M,g} = \mathrm{Lspec}_{M',g'}$). We say that (M, g) is Laplace- (resp. length-) spectrally rigid if any (M', g') to which it is Laplace- (resp. length-)isospectral, is in fact isometric to (M, g).

2.1.3 Curvature

Let (M, g) be a Riemannian manifold of dimension d, and write ∇ Levi-Civita connection for the metric g. The Riemannian curvature tensor is then

$$\mathrm{Riem} : \Gamma(TM) \times \Gamma(TM) \to \Gamma(\mathrm{End}\, TM) \tag{2.2}$$

$$\mathrm{Riem}(X, Y) : \left(Z \mapsto \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\right). \tag{2.3}$$

Let P be a plane in T_pM and $u, v \in P$ be a pair of linearly independent vectors, so that $P = \mathbb{R}u + \mathbb{R}v$. Then the ratio

$$K_p(u, v) = \frac{g_p(\mathrm{Riem}_p(u, v)u, v)}{g_p(u, u), g_p(v, v) - g_p(u, v)^2} \tag{2.4}$$

is independent of the choice of basis u, v of P. Thus K_p defines a function on the Grassmanian $G_p^2 M$ of 2-planes in T_p^M, called the **sectional curvature** of (M, g) at p.

For each $p \in M$ and $x, y \in T_pM$, the **Ricci curvature** $\mathrm{Ric}_p(x, y)$ is the trace of the endomorphism $\mathrm{Riem}_p(x, y) \in \mathrm{End}(T_pM)$. If $(e^1, \ldots, e^d)$ is an orthonormal basis of T_pM with respect to g_p, then one has

$$\mathrm{Ric}_p(x, y) = \sum_{j=1}^{d} g_p(\mathrm{Riem}_p(x, e^j)y, e^j). \tag{2.5}$$

The **scalar curvature** is the function $\mathrm{scal} : M \to \mathbb{R}$ whose value at a point $p \in M$ is given by the trace of the Ricci curvature:

$$\mathrm{scal}_p = \sum_{i \neq j} g_p(\mathrm{Riem}(e^i, e^j)e^i, e^j). \tag{2.6}$$

If $x^1, ..., x^d$ is a coordinate system about p, one has

$$\mathrm{Riem}_q\left(\frac{\partial}{\partial x^h}, \frac{\partial}{\partial x^i}\right)\frac{\partial}{\partial x^j} = \sum_l \mathrm{Riem}^l_{jhi}(q) \frac{\partial}{\partial x_l}. \tag{2.7}$$

for a collection of real valued smooth function Riem^l_{jhi} of $x^1, ..., x^d$. Then one has the following expressions for the Ricci curvature

$$\mathrm{Ric}_{ij} := \mathrm{Ric}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}) = \sum_m \mathrm{Riem}^m_{ijm}, \tag{2.8}$$

and scalar curvature

$$\mathrm{scal} = \sum_{i,j} g^{ij}\,\mathrm{Ric}_{ij} \tag{2.9}$$

where g^{ij} is the inverse matrix to $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$.

We define smooth real valued functions on M locally by the formulae,

$$|\,\mathrm{Riem}\,|^2(q) = \sum_{i,j,k,l} (\mathrm{Riem}^l_{ijk}(q))^2 \tag{2.10}$$

$$|\,\mathrm{Ric}\,|^2(q) = \sum_{i,j} (\mathrm{Ric}_{ij}(q))^2. \tag{2.11}$$

Proposition 2.1.1. Berger.Gauduchon.Mazet-[SpectreVarieteRiemannienne]1971 *Let (M, g) be a Riemannian manifold of dimension d.*

1. *For all $p \in M$, one has*

$$|\,\mathrm{Ric}\,|^2_p \geq \frac{|\,\mathrm{scal}_p\,|^2}{d} \tag{2.12}$$

with equality (for all $p \in M$) if and only if $\mathrm{Ric} = \frac{\mathrm{scal}}{n} \cdot g$.

2. *For all $p \in M$, one has*

$$|\,\mathrm{Riem}\,|^2_p \geq 2\frac{|\,\mathrm{Ric}\,|^2_p}{d-1} \tag{2.13}$$

with equality (for all $p \in M$) if and only if (M, g) has constant sectional curvature.

2.1.4 Heat kernel

The heat **heat kernel** on M is the fundamental solution to the **heat equation** on M. That is, it is the unique smooth function $K_M : M \times M \times \mathbb{R}_{>0} \to \mathbb{R}$ such that: Given any initial data $f : M \to \mathbb{R}$, the solution $F : M \times \mathbb{R}_{>0} \to \mathbb{R}$ of the heat equation

$$\Delta F = -\frac{\partial F}{\partial t} \tag{2.14}$$

$$F(x,0) = f(x) \tag{2.15}$$

is given by

$$F(x,t) = \int_M K_M(x,y,t) f(x) \, \mathrm{d}v_g(x) \tag{2.16}$$

By a theorem of Minakshisundaram and Pleijel **Minakshisundaram.Pleijel-[PropertiesEigenfunct** there exist a sequence of functions $u^{(k)}_{M,g} : M \to \mathbb{R}$ such that for each $x \in M$, the value $u^{(k)}_{M,g}(x)$ is given by universal formulae in terms of the curvature tensor of M and its covariant derivatives at x such that

$$K_M(x,x,t) \sim \frac{1}{(4\pi t)^{\dim M/2}} \sum_{k=0}^{\infty} u^{(k)}_{M,g}(x) t^k t, \qquad \text{as } t \to 0^+. \tag{2.17}$$

In particular, **Berger-[PanoramicViewRiemannian]2003** one has

$$u^{(0)}_{M,g}(x) = 1 \tag{2.18}$$

$$u^{(1)}_{M,g}(x) = \frac{1}{6} \operatorname{scal}(x) \tag{2.19}$$

$$u^{(2)}_{M,g}(x) = \frac{1}{360} \left(2|\operatorname{Riem}|^2_x - 2|\operatorname{Ric}|^2_x + 5 \operatorname{scal}^2(x) \right) \tag{2.20}$$

where scal, Ric, and Riem are the scalar, Ricci, and Riemannian curvature tensors. As the heat kernel $K_{M,g}(x,y,t)$ itself depends only on the the Riemannian structure (M,g), so too do the numbers

$$a^{(k)}_{M,g} := \int_M u^{(k)}_{M,g}(x)\,\mathrm{d}v_g(x), \tag{2.21}$$

for each $k \geq 0$. We refer to $a^{(k)}_{M,g}$ as the k-th **heat invariant** of (M,g).

It turns out that the heat invariants actually depend only on the Laplace-spectrum of (M,g), as the following arguments show.

Proposition 2.1.2. *Suppose (M,g) and (M',g') are isospectral closed Riemannian manifolds. Then for all $t > 0$, one has $\int_M K_M(x,x,t)\,\mathrm{d}v_g(x) = \int_{M'} K_{M'}(x,x,t)\,\mathrm{d}v_{g'}(x)$.*

Proof. Let let $\{\lambda_k : k \geq 0\}$ be the common sequence $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$ of Laplace eigenvalues for (M,g) and (M',g') and pick an orthonormal sequence of eigenfunctions φ_k on M (resp. φ'_k on M') with eigenvalues λ_k. Then one can express the heat kernels for (M,g) and (M',g') as

$$K_{M,g}(x,y,t) = \sum_{k\geq 0} e^{-t\lambda_k}\varphi_k(x)\varphi_k(y) \tag{2.22}$$

$$K_{M',g'}(x,y,t) = \sum_{k\geq 0} e^{-t\lambda_k}\varphi'_k(x)\varphi'_k(y) \tag{2.23}$$

with rapid convergence, uniformly in $x,y \in M$ (resp. in M'). Integrating along the diagonal and passing the integral through the sum, and using the fact that each φ_k has unit $L^2(M,v_g)$ norm, we find

$$\int_M K_{M,g}(x,y,t)\,\mathrm{d}v_g = \sum k \geq 0 e^{-t\lambda_k} \int \varphi_k^2(x)\,\mathrm{d}v_g = \sum_{k\geq 0} e^{-t\lambda_k}. \tag{2.24}$$

Carying out the same computation for (M',g'), we conclude

$$\int_M K_{M,g}(x,x,t)\,\mathrm{d}v_g = \sum_{k\geq 0} e^{-t\lambda_k} = \int_{M'} K_{M',g'}(x,x,t)\,\mathrm{d}v_{g'}. \tag{2.25}$$

□

Corollary 2.1.1. *If (M, g) and (M', g') are isospectral then for all $k \geq 0$, one has*

$$\int_M u_{M,k}(x)\,\mathrm{d}v_g(x) = \int_{M'} u_{M',k}(x)\,\mathrm{d}v_{g'}(x). \tag{2.26}$$

In particular: the dimension, volume, and total scalar curvature for (M, g) and (M', g') coincide.

Proposition 2.1.3. Berger.Gauduchon.Mazet-[SpectreVarieteRiemannienne]1971 *Suppose (M, g) and (M', g') are isospectral closed Riemannian manifolds, and that (M, g) is a surface with constant scalar curvature κ. Then (M', g') is also a surface with constant scalar curvature κ. Furthermore, M and M' are homeomorphic.*

Proof. For any Riemannian 2-manifold (M', g'), one has $|\operatorname{Ric}|^2 = \frac{\operatorname{scal}^2}{2}$ and $|\operatorname{Riem}|^2 = 2|\operatorname{Ric}|^2 = \operatorname{scal}^2$. Thus, the universal expression for the second term in the heat kernel asymptotic expansion is

$$\frac{1}{360}\left(2|\operatorname{Riem}|^2 - |\operatorname{Ric}|^2 + 5\operatorname{scal}^2\right) = \frac{1}{60}\operatorname{scal}^2. \tag{2.27}$$

Furthermore, by Cauchy-Schwarz (in $L^2(M', v_{g'})$):

$$\left(\int_{M'} \operatorname{scal}_{g'}\,\mathrm{d}v_{g'}\right)^2 \leq \left(\int_{M'} \operatorname{scal}_{g'}^2\,\mathrm{d}v_{g'}\right)\left(\int_{M'} 1\,\mathrm{d}v_{g'}\right) \tag{2.28}$$

with equality if and only if $\operatorname{scal}_{g'}$ and 1 are linearly dependent as functions on M', which is to say: if and only if $\operatorname{scal}_{g'}$ is constant. Now note that, as

$$\int_{M'} \operatorname{scal}_{g'}\,\mathrm{d}v_{g'} = \int_M \operatorname{scal}_g\,\mathrm{d}v_g, \tag{2.29}$$

$$\int_{M'} \operatorname{scal}_{g'}^2\,\mathrm{d}v_{g'} = \int_M \operatorname{scal}_g^2\,\mathrm{d}v_g \tag{2.30}$$

$$\int_{M'} 1\,\mathrm{d}v_{g'} = \int_M 1\,\mathrm{d}v_g, \tag{2.31}$$

and by our assumption that scal_g is constant on M, the inequality in 2.28 is an equality. Thus, the scalar curvature on (M', g') is constant, and is readily seen to be equal to that on (M, g). $\square$

Proposition 2.1.4. Berger.Gauduchon.Mazet-[SpectreVarieteRiemannienne]1971 *Let* (M, g) *be a closed Riemannian manifold of dimension* 3 *with constant sectional curvature* σ*, and suppose* (M', g') *is isospectral to* (M, g)*. Then* $\dim M' = 3$ *and* (M', g') *has constant sectional curvature* σ*.*

2.2 Quaternion algebras

A **quaternion algebra** over a field k is a central simple k-algebra of dimension 4. For any field k, an example of a quaternion algebra is the collection of 2×2 matrices with entries in k: $\mathrm{M}_2(k)$. We refer to $\mathrm{M}_2(k)$ as the **split quaternion algebra** over k. In any case, a quaternion algebra A over k is canonically equipped with a **reduced trace** $\mathrm{trd}_A : A \to k$, **reduced norm** $\mathrm{nrd}_A : A \to k$, and **involution** $(\cdot)^*$ which are related by the formulae:

$$\mathrm{trd}_A(x) = x + x^*, \quad \mathrm{nrd}_A(x) = xx^*, \quad \text{for all } x \in A, \tag{2.32}$$

and each $x \in A$ satisfies its **characteristic polynomial**:

$$p(X; x) := X^2 - \mathrm{trd}_A(x)X + \mathrm{nrd}_A(x) \in k[X]. \tag{2.33}$$

In equations 2.32 we are identifying the center of A with k. Furthemore, trd_A is k-linear, nrd_A is multiplicative, and $(\cdot)^*$ fixes the center k of A. The second expression in 2.32 reveals that A is not a division algebra, if and only if there exists a nonzero elementwith reduced norm zero. Indeed, so long as $\mathrm{nrd}(x)$ is nonzero, one has $x^{-1} = (\mathrm{nrd}\, x)^{-1}x^*$. Regarding A as a 4-dimensional vectorspace over k the reduced norm is a quadratic form, and we have

Proposition 2.2.1. *A quaternion algebra* A *over a field* k *is a division algebra if and only if* $\mathrm{nrd}_A : A \to k$ *is anisotropic. In particular, if* k *is an algebraically closed field then no quaternion algebra over* k *is a division algebra.*

In fact, one can say more:

Proposition 2.2.2. *If* k *is algebraically closed, then every quaternion algebra over* k *is isomorphic to* $\mathrm{M}_2(k)$*.*

The reduced trace and norm maps are invariant under $A^\times$ conjugation, and as every automorphism of a central simple algebra is inner (by Artin Wedderburn [TODO: cite!]), we have

Proposition 2.2.3. *Let A, B be quaternion algebras over fields k and ℓ respectively, and suppose that $\iota : k \subset \ell$ is a field embedding. Suppose $\varphi : A \to B$ is a k-algebra homomorphism. Then one has*

$$\mathrm{trd}_B(\varphi(x)) = \iota(\mathrm{trd}_A(x)) \quad \textit{and} \quad \mathrm{nrd}_B(\varphi(x)) = \iota(\mathrm{nrd}_A(x)), \quad \textit{for all } x \in A. \tag{2.34}$$

Furthermore for any field extension ℓ/k, the extension of scalars $A \otimes_k \ell$ is itself a quaternion algebra over ℓ. Combining this with Proposition 2.2.3 we have

Proposition 2.2.4. *If A is a quaternion division algebra over a field k, then there exists an extension ℓ of k such that $A \otimes_k \ell \cong \mathrm{M}_2(\ell)$.*

If ℓ is an extension of k as in proposition 2.2.4, then we say that A **splits** over ℓ. It is the case that a splitting field for A can always be taken to be quadratic over k.

2.2.1 Over local fields

Suppose now that k is a local field of characteristic zero, so that k is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or a finite extension of $\mathbb{Q}_p$ for some prime p. In this case, quaternion algebras over k are classified by the following

Proposition 2.2.5. Maclachlan.Reid-[ArithmeticHyperbolic3Manifolds]2003 *Let k be a local field of characteristic zero. If $k = \mathbb{C}$, then any quaternion algebra over k is isomorphic to $\mathrm{M}_2(\mathbb{C})$. If $k = \mathbb{R}$, then any quaternion algebra over k is isomorphic to either $\mathrm{M}_2(\mathbb{R})$ or the division algbebra:*

$$D = \{ \left(\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix}\right) : a, b \in \mathbb{C}\} \subset \mathrm{M}_2(\mathbb{C}). \tag{2.35}$$

If k is a finite extension of $\mathbb{Q}_p$, then any quaternion algebra over k is isomorphic to either $\mathrm{M}_2(k)$ *or*

$$D = \{ \left(\begin{smallmatrix} a & b \\ \varpi b' & a' \end{smallmatrix} \right) : a, b \in K \} \subset \mathrm{M}_2(K) \tag{2.36}$$

where K is the unique unramified quadratic extension of k, $\varpi \in k$ is a uniformizer, and $\cdot'$ is the nontrivial automorphism of K/k.

We remark that the quaternion algebra D in the above proposition is the unique quaternion algebra over k which is not split over k. We say that A is **ramified** over k if $A \cong D$, and **unramfied** if $A \cong \mathrm{M}_2(k)$. The content of the above proposition is that every quaternion algebra over k is either ramified or unramified.

2.2.2 Over number fields

Now we suppose k is a number field, and R its ring of integers. We write Ω_k for its set of places, and write $\nu \in \Omega_k^\infty$ (resp. $\nu \in \Omega_k^f$) if ν is archimedian (resp. nonarchimedian). We identify nonarchimedian places $\nu \in \Omega_k^f$ with discrete valuations $\nu : k \to \mathbb{Z}$ on k and in turn, prime ideals $\mathfrak{p} = \{x \in k : \nu(x) > 0\}$ in R. For a place $\nu \in \Omega_k$ we write k_ν for the corresponding local field, and if $\nu \in \Omega_k^f$ corresponding to a prime ideal $\mathfrak{p}$ of R we write R_ν or $R_\mathfrak{p}$ for the closure of R in k_ν. We say $\nu \in \Omega_k^\infty$ is real (resp. complex) and write $\nu \in \Omega_k^{\infty,\mathbb{R}}$ (resp. $\Omega_k^{\infty,\mathbb{C}}$) if $k_\nu \approx \mathbb{R}$ (resp. $\mathbb{C}$).

If A is a quaternion algebra over k, then for each place ν, the embedding $k \hookrightarrow k_\nu$ of fields induces an embeding $A \hookrightarrow A_\nu := A \otimes_k k_\nu$ of A into a quaternion algebra A_ν over k_ν, in accordance with Proposition 2.2.4.

Following the terminology of proposition 2.2.5, we say that A is **ramified** over a place $\nu \in \Omega_k$ if A_ν is ramified over k_ν, and **unramified** otherwise.

The following theorem asserts that quaternion algebras (or more generally, central simple algebras) over a number field satisfy a local-global principle, as follows:

Proposition 2.2.6. *Suppose A, A' are two quaternion algebras over a common number field k. Then $A \cong A'$ if and only if, for each place $\nu \in \Omega_k$ the local completions A_ν and A'_ν are isomorphic.*

For each $\nu \in \Omega_k$, there are only two possiblities for A_ν up to isomorphism, so the theorem amounts to the assertion: that two quaternion algebras over a common number field are isomorphic if and only if they ramify over the same set of places.

By global class field theory, the set $\operatorname{Ram} A$ of places over which A ramifies is a finite set with even cardinality. Conversely, given any finite set $S \subset \Omega_k$ of even cardinality and such that $S \cap \Omega_k^{\infty,\mathbb{C}} = \emptyset$ there is a unique quaternion algebra A with $\operatorname{Ram} A = S$, up to isomorphism. A is a division algebra if and only if $\operatorname{Ram} A$ is nonempty, wheras $\operatorname{Ram} A = \emptyset$ if and only if A is isomorphic to $\mathrm{M}_2(k)$. Thus

Proposition 2.2.7. *Let k be a number field. Then there is a bijection:*

$$\{\textit{quaternion algebras over } k\} \leftrightarrow \{\textit{finite subsets of } \Omega_k \setminus \Omega_k^{\mathbb{C}} \textit{ of even cardinality}\}$$

given by sending $A \mapsto \operatorname{Ram} A$.

2.3 Quaternion orders

2.3.1 Local orders

In this section, k is a nonarchimedean local field of characteristic zero, with ring of integers R having maximal ideal $\mathfrak{p}$. If A is an algebra over k, then $x \in A$ is an **integer** (or is integral) if the the algebra it generates over R is finitely generated as an R-module. An R-**order** (or simply an order) in A is a subset $\mathcal{O}$ satisfying the following properties:

1. $\mathcal{O}$ is a subring of A containing 1;

2. $\mathcal{O}$ is a finitely generated R-submodule of A;

3. $\mathcal{O}$ generates A as a k-algebra.

We say an order $\mathcal{O}$ in A is **maximal** provided it is not contained in any other order. If A is finitely generated as k-algebra, then it contains a maximal order, and any order is contained in a maximal one.

It follows immediately from the definition that every element of an order is an integer. Conversely every integer is contained in *some* maximal order, although in general, the set of all integers in A need not be an order. Concerning quaternion algebras, however, we have

Proposition 2.3.1. Maclachlan.Reid-[ArithmeticHyperbolic3Manifolds]2003 *Let A be a quaternion algebra over a nonarchimedean local field k. Then the set of all integers in A is an order in A if and only if A is ramified. In this case, it is the unique maximal order in A.*

Concerning the unramified case, we have

Proposition 2.3.2. Maclachlan.Reid-[ArithmeticHyperbolic3Manifolds]2003 *Let L be an R-lattice in k^2. Then the set*

$$\mathrm{End}(L) = \{x \in \mathrm{M}_2(k) : xL \subset L\} \tag{2.37}$$

is a maximal order in $\mathrm{M}_2(k)$*. Any maximal order in* $\mathrm{M}_2(k)$ *is equal to* $\mathrm{End}(L)$ *for some R-lattice L in k^2. Moreover, if L and L' are R-lattices in k^2, then* $\mathrm{End}(L) = \mathrm{End}(L')$ *if and only if $L = \lambda L'$ for some $\lambda \in k^\times$.*

As R is a principal ideal domain, every R-lattice in k^2 is free of rank 2, so that any R-lattice L in k^2 is isomorphic to the standard R-lattice R^2. Further, any isomorphsim $g : L \to R^2$ extends uniquely to an automorphism $g : L \otimes_R k = k^2 \to k^2$. Then, as $\mathrm{M}_2(R) = \mathrm{End}(gL) = g\,\mathrm{End}(L)g^{-1}$, we have the following

Corollary 2.3.1. *All maximal orders in* $\mathrm{M}_2(k)$ *are conjugate under* $\mathrm{GL}_2(k)$*.*

As the set of $g \in \mathrm{GL}_2(k)$ such that $g\,\mathrm{M}_2(R)g^{-1} = \mathrm{M}_2(R)$ is $k^\times\,\mathrm{GL}_2(R)$, there is a bijection

$$\{\text{maximal orders in } \mathrm{M}_2(k)\} \leftrightarrow \mathrm{GL}_2(k)/k^\times\,\mathrm{GL}_2(R). \tag{2.38}$$

Remark 2.3.1. The set of maximal orders in $\mathrm{M}_2(k)$ for k a nonarchimedean local field can be equipped with an adjacency relation, which makes this set into a $q+1$ regular tree, which is known as the **Bruhat-Tits tree** $\mathcal{T}_k$ of $\mathrm{SL}_2(k)$. The tree $\mathcal{T}_k$ serves as a nonarchimedean analogue of the hyperbolic plane, and is an interesting object of study in its own right.

Combining corollary 2.3.1 and proposition 2.3.1, we have

Proposition 2.3.3. *Let k be a nonarchimedean local field, and A a quaternion algebra over A, then there is a unique conjugacy class of maximal order in A.*

2.3.2 Global orders

Now suppose that k is a number field, and resume the notation of 2.2.2. As in the local setting, an R-order in a k-algebra A is an R-subalgebra which contains a k-basis for A.

Let A be a quaternion algebra over k, and $\mathcal{O}$ an order in A. Then for each finite place ν, the localization $\mathcal{O}_\nu = \mathcal{O} \otimes_R R_\nu$ is an R_ν order in A_ν, and if $\mathcal{O}$ is a maximal order in A then for each ν, $\mathcal{O}_\nu$ is a maximal order in A_ν. Conversely:

Proposition 2.3.4. *Let $\mathcal{O}$ be an order in a quaternion algebra A over a number field ν such that for every finite place ν, the localization $\mathcal{O}_\nu$ is a maximal order in A_ν. Then $\mathcal{O}$ is a maximal order in A.*

As in the local setting, maximal orders exist in any quaternion algebra. The group $A^\times$ acts on the set of maximal orders by conjugation. The number of orbits of this action is finite, and is called the **type number** of A. The type number of A is denoted t_A.

In much the same way that the class number $h_k = I_k/P_k$ of a number field k measures the failure of the local-global-principle for principality of ideals, the type number t_A measures the failure of the local-global-principle for conjugacy of maximal orders. If A satisfies the so-called **Eichler conidtion**, that A is unramified over at least one archimedean place, then these two quantities are explicitly related:

Proposition 2.3.5. Maclachlan.Reid-[ArithmeticHyperbolic3Manifolds]2003 *Let A be a quaternion algebra over a number field k satisfying the Eichler condition. Then the type number of A is the order of the quotient group of the restricted class group of k by the* *subgroup generated by the classes of the prime ideals of R over which A is ramified and the squares of prime ideals of R.*

A consequence of this fact that will be crucial to a latter argument is

Corollary 2.3.2. *Suppose k is a number field with class number 1. Then any quaternion algebra A over k that satisfies the Eichler condition has type number 1. More generally, the conclusion holds provided k has odd class number.*

2.4 Units of quaternion algebras

If A is a quaternion algebra over a field k, we write

$$A^1 = \{a \in A \mid \operatorname{nrd}_A a = 1\} \tag{2.39}$$

for the subgroup of $A^\times$ consisting of elements of reduced norm 1. If k is a number field, and $\nu \in \Omega_k$ is a place over which A is unramified, then the natural inclusion $k \to k_\nu$ induces an inclusion $A^1 \hookrightarrow A^1_\nu$. Picking an isomorphism $A_\nu \cong \mathrm{M}_2(k_\nu)$, we can identify A^1_ν with $\mathrm{SL}_2(k_\nu)$, and thus A^1 with a subgroup of $\mathrm{SL}_2(k_\nu)$.

For a nonempty finite set $S \subset \Omega_k$ of places, write

$$k_S = \prod_{\nu \in S} k_\nu, \quad k^S_{\mathcal{A}} = \prod_{\nu \notin S}{}' k_\nu \tag{2.40}$$

for finite product of fields, and for the S-adeles of k respectively; we use coordinate embeddings to view each as subrings of the full adeles $k_{\mathcal{A}}$.

Under the diagonal embedding, it is well known that k embeds as a discrete, cocompact lattice in $k_{\mathcal{A}}$. The classical weak/strong approximation theorems concern the projections from $k_{\mathcal{A}}$ to k_S and $k^S_{\mathcal{A}}$ respectively, under the the diagonal embedding $k \to k_{\mathcal{A}}$.

Let G be a linear algebraic group over k, and for any commutative k-algebra E, we write $G(E)$ for the points of G with values in E. The embeddings $k \to k^S_{\mathcal{A}}$ and $k \to k^S_{\mathcal{A}}$ and their product $k \to k_{\mathcal{A}}$ induce embeddings $G(k) \to G(k^S_{\mathcal{A}})$, $G(k) \to G(k_S)$, and $G(k) \to G(k_{\mathcal{A}})$.

We say that G satisfies **weak approximation** (resp. **strong approximation**) at S if the natural map $G(k) \to G(k_S)$ (resp. $G(k) \to G(k_{\mathcal{A}}^S)$) has dense image. We say that G satisfies **weak approximation** (resp. **strong approximation**) if it satisfies weak approximation (resp. strong approximation) at every finite set $S \subset \Omega_k$ of places.

Remark 2.4.1. The classical weak/strong approximation theorems can be understood as the assertion that the additive group $\mathbb{G}_a$, as a linear algebraic group over k, satisfies weak/strong approximation.

Consider now the linear algebraic group G over a number field k associated with the units of reduced norm one in a quaternion algebra A over k. Then for all $\nu \notin \mathrm{Ram}(A)$ we have $G(k_\nu) = A_\nu^1 \cong \mathrm{SL}_2(k_\nu)$, and if $\nu \in \mathrm{Ram}(A)$, then $G(k_\nu) = D_\nu^1$ is compact.

We will make use of the following form of strong approximation:

Proposition 2.4.1. Maclachlan.Reid-[ArithmeticHyperbolic3Manifolds]2003 *Let A be a quaternion algebra over a number field k, and let S be a finite set of places of k such that $S \cap \Omega_k^\infty \neq$ and for at least one $\nu_0 \in S$, $\nu_0 \notin \mathrm{Ram}(A)$. Then $A_k^1 A_S^1$ is dense in $A_{\mathcal{A}}^1$.*

2.5 SL_2

Locally, the group of units in a quaternion algebra is isomorphic to SL_2 almost everywhere. In this section, we review the structure theory of SL_2 over fields, and then over local rings, which we will directly use in the main theorem of this paper.

2.5.1 Over a field

To begin, we let Ω be an algebraically closed field of characteristic $p \geq 0$. Define

$$\mathrm{SL}_2(\Omega) = \{g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\Omega) \mid \det(g) = ad - bc = 1\}. \tag{2.41}$$

Let $u : \Omega \to \mathrm{SL}_2(\Omega)$ be the homomorphism $b \mapsto u(b) = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$. Write U for its image, and set $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Then the set $U \cup \{w\}$ generates $\mathrm{SL}_2(\Omega)$. In particular, for each $b \in \Omega^\times$ one has

$$s(b) := wu(b^{-1})wu(b)wu(b^{-1}) = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \tag{2.42}$$

so that s is a-posteriori a homomorphism $\Omega^\times \to \mathrm{SL}_2(\Omega)$ (e.g. a co-character). Write A for its image.

The relations among u, s, and w are given by the formulae:

1. $w^2 = s(-1)$

2. $u(b)u(b') = u(b+b')$ for all $b, b' \in \Omega$

3. $s(a)s(a') = s(aa')$ for all $a, a' \in \Omega^\times$

4. $s(a)u(b)s(a^{-1}) = u(ba^2)$ for all $b \in \Omega$ and $a \in \Omega^\times$.

An immediate consequence of last item above is that A normalizes U, so that the set $B = AU$ is in fact a subgroup of $\mathrm{SL}_2(\Omega)$ and as $A \cap B = \{\mathrm{id}\}$, the map $A \times U \to B$ sending $(a, b) \mapsto s(a)u(b) = \left(\begin{smallmatrix} a & ab \\ 0 & 1/a \end{smallmatrix}\right)$ is a bijection.

One has the **Bruhat decomposition**

$$\mathrm{SL}_2(\Omega) = BwB \sqcup B \tag{2.43}$$

The big cell BwB consists of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $c \neq 0$, and constitutes a single (left) U orbit. Thus $BwB = UwB$, and the map $U \times A \times U$ given by $(b', a, b) \mapsto u(b')ws(a)u(b) = \left(\begin{smallmatrix} -ab' & a^{-1}(1+a^2bb') \\ -a & -ab \end{smallmatrix}\right)$ is a bijection.

Proposition 2.5.1. Suzuki-[GroupTheory]2014 *Suppose that Ω is an algebraically closed field of characteristic $p \geq 0$. Then*

1. *The center Z of $\mathrm{SL}_2(\Omega)$ is $\{\mathrm{id}\}$ if $p = 2$ and otherwise is cyclic of order 2, generated by $s(-1) = -\mathrm{id}$.*

2. *Every conjugacy class in $\mathrm{SL}_2(\Omega)$ has a representative of the form $s(a) \in A$ for some $a \in \Omega^\times$ or $\pm u(b) \in ZU$ for some $b \in \Omega$.*

3. *if $p \neq 2$ then $\mathrm{SL}_2(\Omega)$ contains a unique element of order 2*

4. *Let $x \in \mathrm{SL}_2(\Omega)$ be an element with finite order n. Then x is conjugate to $\pm u(b)$ for some nonzero $b \in \Omega$ if and only if $p > 0$ and $p|n$. In this case, x is either p or $2p$.*

5. *The normalizer of U in $\mathrm{SL}_2(\Omega)$ is B. If conjugation by $x \in B$ fixes any nonidentity element of U, then $x \in ZU$.*

6. *The centralizer in $\mathrm{SL}_2(\Omega)$ of any nonscalar element of A is A and the normalizer $N(A)$ in $\mathrm{SL}_2(\Omega)$ of any subgroup A' of A containing at least 3 element is $N(A) = \langle A, w \rangle$.*

7. *The centralizer of any nonscalar element of $\mathrm{SL}_2(\Omega)$ is abelian.*

Let F be a subfield of Ω. Then there is a natural inclusion $\mathrm{SL}_2(F) \to \mathrm{SL}(2, \Omega)$, and the homomorphisms s, u restrict to $F^\times, F$, mapping to subgroups $A(F), U(F) \le \mathrm{SL}_2(F)$ and $B(F) = A(F)U(F)$.

Projective spaces:

We recall that n-dimensional projective space over Ω is the set $\mathbb{P}^n_\Omega$ of 1-dimensional Ω-linear subspaces in Ω^{n+1}, and that the linear action of $\mathrm{GL}(n+1, \Omega)$ on Ω^{n+1} induces one of $\mathrm{PGL}(n+1, \Omega) = \Omega^\times \backslash \mathrm{GL}(n+1, \Omega)$ on $\mathbb{P}^n_\Omega$. For any subfield F of Ω one can similarly define $\mathbb{P}^n_\Omega$, on which $\mathrm{PGL}(n+1, F)$ naturally acts. In particular, the **projective line** over F is the quotient of $F^2 \setminus \{0\}$ by the relation $v \sim v'$ if $v = \lambda v'$ for some $\lambda \in F^\times$. If $v \in F^2 \setminus \{0\}$, then we write $[v]$ for its equivalence class in $\mathbb{P}^1(F)$, and if $v = ae_1 + be_2$ then we write $[v] = [a : b]$ and refer to the pair (a, b) as **homogeneous** coordinates for $[v]$. We identify F with a subset of $\mathbb{P}^1(F)$ via $x \mapsto [x : 1]$. The complement of F in $\mathbb{P}^1(F)$ is the single point $[0 : 1] = [e_2]$ which we denote by ∞, so that $\mathbb{P}^1 = F \cup \{\infty\}$.

In this model, the action of $\mathrm{GL}(2, F)$ on $\mathbb{P}^1_F$ is given by fractional linear transformations: for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, F)$ one has

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \begin{cases} \frac{az+c}{cz+d}, & z \in F \\ d, & z = \infty. \end{cases} \tag{2.44}$$

Proposition 2.5.2. *Let F be a subfield of the algebraically closed field Ω.*

- *The stabilizer of ∞ in $\mathrm{SL}_2(F)$ is $B(F)$ and the stabilizer of 0 is $wB(F)w^{-1}$.*

- *The pointwise stabilizer of the pair* $\{0, \infty\}$ *is* $B(F) \cap wB(F)w^{-1} = A(F)$*, while the setwise stabilizer is its normalizer* $N(F) := \langle A(F), w \rangle$*.*
- *The subgroup* $U(F)$ *acts simply transitively on the set* $F = \mathbb{P}^1(F) \setminus \infty$*.*
- *The orbits of* $A(F)$ *on* $F^\times = \mathbb{P}^1(F) \setminus \{0, \infty\}$ *are in bijection with square classes* $F^\times/(F^\times)^2$ *in* $F^\times$*.*
- *A noncentral element* g *of* $\mathrm{SL}_2(F)$ *fixes at most two fixed points in* $\mathbb{P}^1_F$*.*

We say that g is F-**hyperbolic**, F-**parabolic**, or F-**elliptic** if g fixes 2, 1, or 0 points in $\mathbb{P}^1_F$, and refer to this descriptor as the F-type of g. It is clear that the F-type of an element depends only on its $\mathrm{SL}_2(F)$ (or even $\mathrm{GL}(2, F)$)) conjugacy class.

For an element $x = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{M}_2(F)$, we write

$$p(X; x) := \det(XI_2 - x) = X^2 - \operatorname{tr} x X + \det x \in F[X] \tag{2.45}$$

for its characteristic polynomial and

$$\delta_x := \operatorname{tr}^2 x - 4 \det x \in F \tag{2.46}$$

for its descriminant. Note that δ_x is invariant under conjugation by $\mathrm{GL}(2, F)$, and that $p(X; x)$ has a pair of distinct roots in F if and only if δ_x is a square in $F^\times$. We write $F[x]$ for the subalgebra of $\mathrm{M}_2(F)$ generated by x, which is isomorphic to $F[X]/(p(X; x))$ and if x is noncentral, coincides with the centralizer of x in $\mathrm{M}_2(F)$,

The following proposition relates the F-type of an element to other invariants of a conjugacy class.

Proposition 2.5.3. *Let g be a noncentral element of $\mathrm{SL}_2(F)$. The following equivalencas hold:*

$$g \text{ is } F\text{-hyperbolic} \iff \begin{cases} g \text{ is conjugate in } \mathrm{SL}_2(F) \text{ to an element of } A(F) \\ p(x;g) := x^2 - \operatorname{tr} gx + 1 \text{ has a pair of distinct roots in } F \\ \delta_g := \operatorname{tr}^2 g - 4 \text{ is a square in } F^\times \\ F[g] \cong F \times F \text{ as } F \text{ - algebras} \end{cases} \tag{2.47}$$

$$g \text{ is } F\text{-parabolic} \iff \begin{cases} g \text{ is conjugate in } \mathrm{SL}_2(F) \text{ to an element of } ZU(F) \\ p(x;g) := x^2 - \operatorname{tr} gx + 1 \text{ has repeated roots in } F \\ \delta_g := \operatorname{tr}^2 g - 4 = 0 \\ F[g] \cong F[X]/(X^2) \text{ as } F \text{ - algebras} \end{cases} \tag{2.48}$$

$$g \text{ is } F\text{-elliptic} \iff \begin{cases} g \text{ is conjugate in } \mathrm{SL}_2(\Omega) \text{ to an element of } A \setminus A(F) \\ p(x;g) := x^2 - \operatorname{tr} gx + 1 \text{ has no roots in } F \\ \delta_g := \operatorname{tr}^2 g - 4 \text{ is not a square in } F^\times \\ F[g] \cong F[X]/(X^2 - \delta_g) \text{ is a quadratic extension of } F \end{cases} \tag{2.49}$$

Remark 2.5.1. When F is algebraically closed, it is evident that no element is F-elliptic. If F is not algebraically closed and g is F-elliptic, then g is F'-hyperbolic for F' a quadratic extension of F. Namely $F' = F[\sqrt{\delta_g}]$.

Unlike ellipticity and hyperbolicity, parabolicty is an absolute property: if g is F-parabolic, then it is F' parabolic for any subfield or extension F' of F.

We will also make use of the following

Proposition 2.5.4. *Suppose noncentral elements $g, h \in \mathrm{SL}_2(F)$ are F-parabolic elements, having unique fixed points P and Q in $\mathbb{P}^1_F$ respectively. If gh is also F-parabolic, then $P = Q$*

Proof. Arguing by contraposition, suppose g and h have distinct fixed points. As the action of $\mathrm{SL}_2(F)$ is doubly transitive on $\mathbb{P}^1_F$ we may assume that $P = 0$ and $Q = \infty$. Then g

and h take the form $\pm\left(\begin{smallmatrix}1&0\\x&1\end{smallmatrix}\right)$ and $\pm\left(\begin{smallmatrix}1&y\\0&1\end{smallmatrix}\right)$ respectively. Then $gh = \pm\left(\begin{smallmatrix}1&y\\x&1+xy\end{smallmatrix}\right)$, which is not parabolic unless $xy = 0$. But then at least one of g, h is central, a contradiction. $\square$

Corollary 2.5.1. *Any subgroup of* $\mathrm{SL}_2(F)$ *consinsting of* F*-parabolic elements is conjugate to a subgroup of* $ZU(F)$.

2.5.2 The adjoint representation

The Lie algebra of $\mathrm{SL}_2(F)$ can be identified with

$$\mathfrak{sl}_2(F) = \{X \in M(2, F) : \operatorname{tr} X = 0\} \tag{2.50}$$

with Lie bracket given by the commutant $[X, Y] = XY - YX$. The group $\mathrm{SL}_2(F)$ acts on its Lie algebra by conjugation, and we write $\mathrm{Ad} : \mathrm{SL}_2(F) \to \mathrm{GL}(\mathfrak{sl}_2(F))$ for the resulting representation.

The **killing form** is the nondegenerate symmetric bilinear form on $\mathfrak{sl}_2(F)$ given by

$$B(X, Y) = \operatorname{tr} XY, \tag{2.51}$$

and the associated quadratic form is

$$Q(X) = B(X, X) = -\det X. \tag{2.52}$$

For $\delta \in F$, we write $\mathcal{O}(\delta)$ for the fiber

$$\mathcal{O}(\delta) = \{X \in \mathfrak{sl}_2(F) : Q(X) = \delta\}. \tag{2.53}$$

The adjoint representation preserves B:

$$B(\mathrm{Ad}\, gX, \mathrm{Ad}\, gY) = B(X, Y) \quad \text{for all } g \in \mathrm{SL}_2(F),\ X, Y \in \mathfrak{sl}_2(F) \tag{2.54}$$

and consequently preserves the sets $\mathcal{O}(\delta)$.

Concerning the orbits of $\mathrm{SL}_2(F)$ on $\mathfrak{sl}_2(F)$ under Ad, we have

Proposition 2.5.5. *For each nonzero δ, the adjoint action of* $\mathrm{SL}_2(F)$ *on* $\mathcal{O}(\delta)$ *is transitive.*

The **nilpotent cone**

$$\mathcal{N} = \{X \in \mathfrak{sl}_2(F) : X^2 = 0\} \tag{2.55}$$

in $\mathfrak{sl}_2(F)$ is a nondegenerate quadric, and coincides with the fiber $\mathcal{O}(0)$ over 0.

By contrast to fibers over $F^\times$ there are always more than one adjoint orbits in the nilpotent cone, as $0 \in \mathcal{N}$. The situation is described in the following

Proposition 2.5.6. *If F is algebraically closed, then the nonzero elements of the nilpotent cone $\mathcal{N} \subset \mathfrak{sl}_2(F)$ form a single adjoint orbit. If F is any field, then the action of* $\mathrm{SL}_2(F)$ *on the set of F-lines in $\mathcal{N}$ is transtive.*

Proof. If $X \in \mathcal{N}$ then $X^2 = 0$, as an endomorphism of k^2. If X is nonzero, then there exists an $\alpha \in \mathrm{GL}(2, F)$ such that $\alpha X \alpha^{-1} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. If F is algebraically closed, then $\alpha' = (\det \alpha)^{-1/2}\alpha$ lies in $\mathrm{SL}_2(F)$ and does the same.

For the second claim, note that the equation $Q(X, X) = -\det X = 0$ defines a rational normal curve of degree 2 in $\mathbb{P}^2_F = \mathbb{P}(\mathfrak{sl}_2(F))$ on which $\mathrm{SL}_2(F)$ acts a group of projective linear automorphisms. Any such action, so long as it is nontrivial, is transitive. $\square$

2.5.3 Over a local field

In this section R is local ring of characteristic zero with maximal ideal $\mathfrak{p}$, which is complete with respect to the $\mathfrak{p}$-adic topology, F is its field of fractions, and $\mathfrak{f}$ its residue field. We pick once and for all a uniformizer ϖ for $\mathfrak{p}$. We write $\bar{\cdot} : R \to \mathfrak{f}$ for reduction modulo $\mathfrak{p}$, as well its canonical extension to any algebra over R.

The structure theory of SL_2 over a field comes into play in two ways with regard to the structure of $\mathrm{SL}_2(R)$: through its projection to $\mathrm{SL}_2(\mathfrak{f})$ and through its inclusion in $\mathrm{SL}_2(F)$.

For each $n \geq 1$, there is a short exact sequence

$$1 \to \mathrm{SL}_2(R, \mathfrak{p}^n) \to \mathrm{SL}_2(R) \to \mathrm{SL}_2(R/\mathfrak{p}^n) \to 1 \tag{2.56}$$

where

$$\mathrm{SL}_2(R, \mathfrak{p}^n) = \{g \in \mathrm{SL}_2(R) : g \equiv \mathrm{id} \mod \mathfrak{p}^n\} \tag{2.57}$$

is the congruence kernel of level $\mathfrak{p}^n$. The group $\mathrm{SL}_2(R)$ is a maximal compact open subgroup of $\mathrm{SL}_2(F)$, and any maximal compact open subgroup in $\mathrm{SL}_2(F)$ is conjugate to $\mathrm{SL}_2(R)$ by an element of $\mathrm{GL}_2(F)$. The congurence kernels $\mathrm{SL}_2(R, \mathfrak{p}^n)$ constitute a neighbhood basis about the identity.

Proposition 2.5.7. *For each $n \geq 1$, the map $g \mapsto \overline{\varpi^{-n}(g - \mathrm{id})}$ induces an isomorphism $\mathrm{SL}_2(R, \mathfrak{p}^n)/\mathrm{SL}_2(R, \mathfrak{p}^{n+1}) \to \mathfrak{sl}_2(\mathfrak{f})$, where the latter is understood as the underlying abelian group. Furthermore, this isomorphism intertwines the action of $\mathrm{SL}_2(R)$ by conjugation on the former, and the adjoint action on the latter.*

Proof. For the first claim, all that needs to be checked is that the given map is in fact a homomorphism. To this end, suppose $g = \mathrm{id} + \varpi^n X$ and $h = \mathrm{id} + \varpi^n Y$ with $X, Y \in \mathrm{M}_2(R)$. Then

$$gh = \mathrm{id} + \varpi^n(X + Y) + \varpi^{2nXY} \equiv \mathrm{id} + \varpi^n(X + Y) \mod \mathfrak{p}^{n+1} \tag{2.58}$$

as desired. The second claim is self-evident. □

Remark 2.5.2. The proof actually demonstrates that more generally, one has an isomorphism $\mathrm{SL}_2(R, \mathfrak{p}^n)/\mathrm{SL}_2(R, \mathfrak{p}^{2n}) \to \mathfrak{sl}_2(R/\mathfrak{p}^{2n})$ as $\mathrm{SL}(2, R/\mathfrak{p}^{2n})$-modules.

In preparation for the main theorem, we will need to characterize when a subgroup H of $\mathrm{SL}_2(R)$ is conjugate into a subgroup of $\mathrm{SL}_2(R, \mathfrak{p}^n)$ by an element of $\mathrm{SL}_2(F)$ or $\mathrm{GL}_2(F)$. We remark that although $\mathrm{SL}_2(R, \mathfrak{p}^n)$ is normal in $\mathrm{SL}_2(R)$, it is not normal in $\mathrm{SL}_2(F)$. For each n, the set of $\alpha \in \mathrm{SL}_2(F)$ such that $\alpha\,\mathrm{SL}_2(R, \mathfrak{p}^n)\alpha^{-1} \subset \mathrm{SL}_2(R)$ consists of finitely many $\mathrm{SL}_2(R)$ cosets.

The main result of this section is the following

Proposition 2.5.8. *Suppose H is a subgroup of $\mathrm{SL}_2(R)$ satisfying*

$$\delta_h \equiv 0 \mod \mathfrak{p}^{2n}, \qquad \textit{for all } h \in H \tag{2.59}$$

Then H is conjugate in $\mathrm{GL}_2(F)$ to a subgroup of $\mathrm{SL}_2(R, \mathfrak{p}^n)$.

Proof. The argument is by induction. For the base case, suppose $n = 1$ and that $H \leq \mathrm{SL}_2(R)$ is a subgroup satisfying

$$\delta_h = (\operatorname{trd} h)^2 - 4 \equiv 0 \mod \mathfrak{p}^2 \tag{2.60}$$

for all $h \in H$. Then by proposition 2.5.3 $\overline{H}$ consists entirely of $\mathfrak{f}$- parabolic elements, so that by corollary 2.5.1, H is conjugate in $\mathrm{SL}_2(\mathfrak{f})$ to a subgroup of $ZU(\mathfrak{f})$, where $U(\mathfrak{f})$ is the group of upper triangular matrices in $\mathrm{SL}_2(\mathfrak{f})$. As the map $\mathrm{SL}_2(R) \to \mathrm{SL}(2, \mathfrak{f})$ is surjective, there is an element $\alpha \in \mathrm{SL}_2(R)$ which reduces to $\overline{\alpha}$ modulo $\mathfrak{p}$. Thus, after replacing H by an $\mathrm{SL}_2(R)$ conjugate, we may assume that each h in H may be written as

$$h = \pm \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} + \varpi\gamma$$

for some $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{M}_2(R)$.

Since $H \leq \mathrm{SL}(2, R)$, one has $\det h = 1$ for all $h \in H$, thus

$$\begin{aligned} 1 &= \det h \\ &= 1 + (a + d - cx)\varpi + (ad - bc)\varpi^2 \end{aligned}$$

so that

$$(a + d - cx)\varpi + (ad - bc)\varpi^2 = 0. \tag{2.61}$$

By hypothesis, we have $(a+d)\varpi = \operatorname{tr} h \mp 2 \equiv 0 \mod \mathfrak{p}^2$ so that $a+d \in \mathfrak{p}$. Consequently, $xc \in \mathfrak{p}$ and as $\mathfrak{p}$ is prime, it follows that at least one of x, c lies in $\mathfrak{p}$. If $x \in \mathfrak{p}$, then $h \in \mathrm{SL}_2(R, \mathfrak{p})$ already. If this is so for all $h \in H$ then the claim is proven.

Supposing otherwise, there exists a $h = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) + \varpi \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ for which $x \in R^\times$. In this case one has $c \in \mathfrak{p}$. Let $\alpha = \left(\begin{smallmatrix} \varpi & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then

$$\alpha g \alpha^{-1} = \left(\begin{smallmatrix} 1 & \varpi x \\ 0 & 1 \end{smallmatrix}\right) + \varpi \begin{pmatrix} a & \varpi b \\ \varpi^{-1} c & d \end{pmatrix}$$

lies in $\mathrm{SL}_2(R, \mathfrak{p})$, completing the proof of the base case.

Now suppose that $\delta_h = \operatorname{tr}^2 h - 4 \equiv 0 \mod \mathfrak{p}^{2n}$ for every $h \in H$. By inductive hypothesis, after conjugating by an element of $\mathrm{GL}_2(F)$ if necessary, we may assume $H \leq \mathrm{SL}_2(R, \mathfrak{p}^{n-1})$.

Thus, each element $h \in H$ can be written as

$$h = \mathrm{id} + \varpi^{n-1} \gamma \tag{2.62}$$

for some $\gamma \in \mathrm{M}_2(R)$, and the assignment $h \mapsto \overline{\gamma}$ identifies the image of H under reduction mod $\mathfrak{p}^n$ with an additive subgroup of $\mathfrak{sl}_2(\mathfrak{f})$.

Note that since $\operatorname{tr}^2 h - 4 \equiv 0 \mod \mathfrak{p}^{2n}$, one has $\operatorname{tr} \gamma \equiv 0 \mod \mathfrak{p}^{n+1}$. Computing determinants as in 2.62,

$$\begin{aligned} 1 &= \det h \\ &= 1 + \varpi^{n-1} \operatorname{tr} \gamma + \varpi^{2n-2} \det \gamma, \end{aligned}$$

and since $\det h = 1$, we find $\operatorname{tr} \gamma + \varpi^{n-1} \det \gamma = 0$. From $\operatorname{tr} \gamma \equiv 0 \mod \mathfrak{p}^{n+1}$ we find $\varpi^{n-1} \det \gamma \equiv 0 \mod \mathfrak{p}^{n+1}$, so that $\det \gamma \equiv 0 \mod \mathfrak{p}^2$. It follows that the image of H under the composite map $H \to \mathrm{SL}_2(R, \mathfrak{p}^{n-1}) \backslash \mathrm{SL}_2(R, \mathfrak{p}^n) \to \mathfrak{sl}_2(\mathfrak{f})$ is contained in the nilpotent cone $\mathcal{N}(\mathfrak{f})$ of $\mathfrak{sl}_2(\mathfrak{f})$.

By proposition 2.5.6, we may replace H by an $\mathrm{SL}_2(R)$-conjugate so that each $g \in H$ takes the form $g = 1 + \varpi^{n-1}\gamma$ where $\gamma = \left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right) + \varpi\delta$ for some $\delta = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M(2, \mathfrak{k}_{\mathfrak{p}})$. From

$$\begin{aligned} \det\gamma &= \det(\left(\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right) + \varpi\delta) \\ &= \varpi^2 \det\delta - \varpi xc \equiv \varpi xc \mod \mathfrak{p}^2, \end{aligned}$$

we find that $xc \equiv 0 \mod \mathfrak{p}$, so that either x or c is in $\mathfrak{p}$. If $x \in \mathfrak{p}$, then $h \in \mathrm{SL}_2(R, \mathfrak{p}^n)$ already. If this is so for all $h \in H$, then the claim is proven. Otherwise, suppose h has $x \in R^\times$ so that $c \in \mathfrak{p}$. Then, with $\alpha = \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix}$, we find that $\alpha h \alpha^{-1} \in \mathrm{SL}_2(R, \mathfrak{p}^n)$ as claimed.

$\square$

3. SPECTRAL FLEXIBILITY

Given a closed, Riemannian manifold M with associated Laplace-Beltrami operator Δ_M, the operator Δ_M acts on the space of L^2 functions or L^2 k-forms $\Omega^k(M)$. We denote the associated eigenvalue spectrum for the operator $\triangle_M$ acting on $\Omega^k(M)$ by $\mathcal{E}_k(M)$. In the case of $k = 0$, we denote the eigenvalue spectrum by $\mathcal{E}(M)$ and refer to this as the eigenvalue spectrum. The spectrum $\mathcal{E}(M)$ is a well studied analytic invariant of the Riemannian manifold M and is known to determine the dimension, volume, and total scalar curvature. A related geometric invariant is the primitive geodesic length spectrum $\mathcal{L}_p(M)$ of M. Assuming for simplicity that M is negatively curved, each free homotopy class of closed curves on M has a unique geodesic representative. We define $\mathcal{L}_p(M)$ to be the set of lengths (with multiplicity) of each geodesic representative in each free homotopy class.

We denote by $H^k(M, \mathbf{Z})$ the k th singular cohomology group of M with trivial $\mathbf{Z}$-coefficients. Given a finite cover $M' \to M$, we have induced homomorphisms Res: $H^k(M, \mathbf{Z}) \to H^k(M', \mathbf{Z})$ and Cor : $H^k(M', \mathbf{Z}) \to H^k(M, \mathbf{Z})$. For a pair of finite covers $M_1, M_2 \to M$, we say that a morphism $\psi_k : H^k(M_1, \mathbf{Z}) \to H^k(M_2, \mathbf{Z})$ is compatible if the diagram

$$\begin{tikzcd}
& H^k(M,\mathbb{Z}) \arrow[dl, bend right, "\mathrm{Res}"'] \arrow[dr, bend left, "\mathrm{Res}"] & \\
H^k(M_1,\mathbb{Z}) \arrow[ur, bend right, "\mathrm{CoInd}"'] \arrow[rr, bend right, "\psi_k"'] & & H^k(M_2,\mathbb{Z}) \arrow[ul, bend left, "\mathrm{CoInd}"]
\end{tikzcd}$$

commutes. We say that M is **large** if there exists a finite index subgroup $\Gamma_0 \leq \pi_1(M)$ and a surjective homomorphism of Γ_0 to a non-abelian free group. The main theorem of this section is the following:

Theorem 1. *Let M be a closed hyperbolic n-manifold that is large and nonarithmetic. Then for $j \in \mathbb{Z}_{>1}$ there exist pairwise non-isometric, finite Riemannian covers $M_1, \dots, M_j \to M$ such that for all $1 \leq i, i' \leq j$*

1. *M_i **and** $M_{i'}$ **are strongly isospectral**: $\mathcal{E}_k(M_i) = \mathcal{E}_k(M_{i'})$ for all $0 \leq k \leq n$*

2. M_i **and** $M_{i'}$ **are length isospectral**: $\mathcal{L}_p(M_i) = \mathcal{L}_p(M_{i'})$

3. For each k*, there exists an isomorphism* $H^k(M_i, \mathbb{Z}) \to H^k(M_{i'}, \mathbb{Z})$ *which is compatible with the covering maps* $M_i, M_{i'} \to M$.

3.1 Preliminaries

We begin with some background on hyperbolic manifolds, their fundamental groups, and relations in cohomology.

3.1.1 Real and Complex hyperbolic spaces

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. Then $\mathbb{K}$-hyperbolic n-space is the Riemannian symmetric space associated to the group $G_{\mathbb{K}}$ of matrices in $\mathrm{SL}(n+1, \mathbb{K})$ preserving the form

$$B^{n,1}_{\mathbb{K}}(x,y) := \begin{cases} -x_{n+1}y_{n+1} + \sum_{j=1}^{n} x_j y_j & \text{if } \mathbb{K} = \mathbb{R} \\ -w_{n+1}\overline{z_{n+1}} + \sum_{j=1}^{n} w_j \overline{z_j} & \text{is } \mathbb{K} = \mathbb{C}, \end{cases} \tag{3.1}$$

so that

$$G_{\mathbb{K}} = \begin{cases} SO(n,1) & \mathbb{K} = \mathbb{R}, \\ SU(n,1) & \mathbb{K} = \mathbb{C}. \end{cases} \tag{3.2}$$

The group of orientation preserving isometries of $\mathbb{H}^n_{\mathbb{K}}$ is then the adjoint form $PG_{\mathbb{K}} = G_{\mathbb{K}}/Z(G_{\mathbb{K}})$ of $G_{\mathbb{K}}$. For a discrete subgroup $\Gamma \leq \mathrm{Isom}(\mathbb{H}^n_{\mathbb{K}})$, the quotient $\Gamma\backslash\mathbb{H}^n_{\mathbb{K}}$ is a $\mathbb{K}$-hyperbolic n-orbifold, and a manifold provided Γ contains no elements of finite order. We say that Γ is a **lattice** in $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{K}})$ provided the quotient $\Gamma\backslash\mathbb{H}^n_{\mathbb{K}}$ has finite volume, and a **uniform lattice** if it furthermore is compact. For $\mathbb{K} = \mathbb{R}$, one has a converse to this construction: given a complete real hyperbolic $n$manifold M, via the action of $\pi_1(M)$ on the universal cover $\mathbb{H}^n_{\mathbb{R}}$, we obtain an injective homomorphism $\pi_1(M) \to \mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$. If furthermore M has finite volume (resp. is compact) then the image of this representation is a lattice (resp. is a cocompact lattice). As this representation depends on a choice of lift $\tilde{p} \in \mathbb{H}^n_{\mathbb{R}}$ of the base point p, the representation is only unique up to conjugation in $\mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$.

We say that a pair of subgroups $\Gamma, \Gamma' \leq PG_{\mathbb{K}}$ are **commensurable** if the intersection $\Gamma \cap \Gamma'$ has finite index in both Γ and Γ'. The commensurator of Γ in $G_{\mathbb{K}}$ is

$$\mathrm{Comm}(\Gamma) = \left\{ g \in \mathrm{PG} : g^{-1}\Gamma g, \Gamma \text{ are commensurable } \right\}. \tag{3.3}$$

By a celebrated theorem of Margulis **Margulis-[DiscreteSubgroupsSemisimple]1991**, for any lattice $\Gamma \in PG_{\mathbb{K}}$ one has a dichotomy: either

1. $|\,\mathrm{Comm}(\Gamma) : \Gamma|$ is finite, in which case we say Γ is **non-arithmetic**, or
2. $\mathrm{Comm}(\Gamma)$ is dense in $G_{\mathbb{K}}$ (in the analytic topology), in which case we say Γ is **arithmetic**.

Remark 3.1.1. This dichotomy holds for lattices in any semisimple Lie group, and our terminology coincides with any of the more sophisticated definitions of arithmeticity.

When $\Gamma \leq G_{\mathbb{K}}$ is discrete and torsion free, the associated $\mathbb{K}$-hyperbolic manifold $\Gamma \backslash \mathbb{H}^n_{\mathbb{K}}$ is a classifying space for Γ (a $K(\Gamma, 1)$-space) as its universal cover $\mathbb{H}^n_{\mathbb{K}}$ is contractible. We will use this fact to translate claims concerning (co)homology groups for these manifolds into statements about the group co(homology) of their fundamental groups.

3.2 Isomorphisms in group cohomology

Given a commutative ring R with identity, a group G, and a pair of subgroups $P_1, P_2 \leq G$, we say that P_1, P_2 are R-equivalent if $R[G/P_1]$ and $R[G/P_2]$ are isomorphic as left $R[G]$ modules. When P_1 and P_2 are $\mathbb{Q}$ equivalent the triple (G, P_1, P_2) is called a **Gassmann-triple**. For general R, the triple is called an R-Gassmann triple.

In this section, we will summarize some basic results that link the group cohomology of $\mathbb{Z}$-equivalent groups.

3.2.1 (co)Restriction, (co)Induction

Fix a commutative ring R with 1. Given a group G and a subgroup P, we denote by Res_P^G the restriction functor $R[G]-\mathrm{Mod} \to R[H]-\mathrm{Mod}$. Restriction admits left and right adjoints respectively given by **induction** Ind_P^G and **coinduction** coInd_P^G: for an $R[H]$ module A, the underlying R-modules are

$$\mathrm{Ind}_P^G(A) = R[G] \otimes_{R[P]} A, \quad \mathrm{coInd}_P^G(A) = \hom_{R[P]}(R[G], A) \tag{3.4}$$

with $R[G]$ module structure given by the R-linear extension of

$$g \cdot (x \otimes a) = gx \otimes a, \quad (g \cdot \varphi)(x) = \varphi(xg) \tag{3.5}$$

respectively. We will make use of the following

Lemma 3.2.1. *Suppose P is a* finite index *subgroup of G, A is an $R[G]$ module, and B is an $R[H]$ module. Then there are isomorphisms of $R[G]$ modules:*

$$\mathrm{Ind}_P^G(A) \cong \mathrm{coInd}_P^G(A) \tag{3.6}$$

$$A \otimes_{R[G]} \mathrm{Ind}_P^G(B) \cong \mathrm{Ind}_P^G(\mathrm{Res}_G^P(A) \otimes_{R[P]} B) \tag{3.7}$$

$$\mathrm{coInd}_P^G(\mathrm{Res}_P^G(A)) \cong A \otimes_R R[G/P] \tag{3.8}$$

Given a group G with subgroups P_1, P_2, we say that a morphism $\psi_k : H^k\left(P_1, \mathrm{Res}_{P_1}^G(A)\right) \to H^k\left(P_2, \mathrm{Res}_{P_2}^G(A)\right)$ is **compatible** if the diagram

$$\begin{tikzcd}
& H^k(G,A) \arrow[dl, bend right, "\mathrm{Res}"'] \arrow[dr, bend left, "\mathrm{Res}"] & \\
H^k\left(P_1, \mathrm{Res}_{P_1}^G(A)\right) \arrow[ur, bend right, "\mathrm{CoInd}"'] \arrow[rr, bend right, "\psi_k"'] & & H^k\left(P_2, \mathrm{Res}_{P_2}^G(A)\right) \arrow[ul, bend left, "\mathrm{CoInd}"']
\end{tikzcd}$$

commutes.

Lemma 3.2.2. *Let G be a finite group and $P_1, P_2 \leq G$ be $\mathbf{Z}$-equivalent subgroups. Then for any $\mathbf{Z}[G]-$-module A and any nonnegative integer k, there is a compatible isomorphism $H^k\left(P_1, \mathrm{Res}^G_{P_1}(A)\right) \to H^k\left(P_2, \mathrm{Res}^G_{p_2}(A)\right)$.*

Proof. By Shapiro's lemma (see **Brown-[CohomologyGroups]2012**), we have

$$H^k\left(P_i, \mathrm{Res}^G_{P_i}(A)\right) = H^k\left(G, \mathrm{CoInd}^G_{P_i}\left(\mathrm{Res}^G_{P_i}(A)\right).\right)$$

By Lemma 3.2.1 the coefficients for the latter cohomology groups are $A \otimes \mathbf{z}\mathbf{Z}\left[G/P_i\right]$, viewed as $\mathbf{Z}[G]$-modules. Since P_1 and P_2 are $\mathbf{Z}$-equivalent, these coefficient modules are $\mathbf{Z}[G]$-isomorphic. Thus, the right hand side of the equality above is actually independent of i, providing the isomorphism as claimed. Compatibility follows from the naturality of the isomorphism in Shapiro's lemma. Specifically, upon choosing an isomorphism of the $\mathbf{Z}[G]$-modules $\mathbf{Z}\left[G/P_1\right]$ and $\mathbf{Z}\left[G/P_2\right]$, isomorphisms in cohomology groups

$$\begin{gathered} H^k\left(P_1, \mathrm{Res}^G_{P_1}(A)\right) \to H^k\left(G, \mathrm{CoInd}^G_{P_1}\left(\mathrm{Res}^G_{P_1}(A)\right)\right) \to \\ H^k\left(G, \mathrm{CoInd}^G_{P_2}\left(\mathrm{Res}^G_{P_2}(A)\right)\right) \to H^k\left(P_2, \mathrm{Res}^G_{P_1}(A)\right) \end{gathered}$$

are induced by isomorphisms of coefficients. □

Suppose now that Γ is any group, and $\psi : \Gamma \to G$ is a surjective homomorphism, and set $\Gamma_i = \psi^{-1}(P_i)$. Then $\Gamma_1, \Gamma_2 \leq \Gamma$ are $\mathbb{Z}$-equivalent. The following assertions are consequences of lemmas 3.2.1,3.2.2.

Lemma 3.2.3. *Let $\psi : \Gamma \to G$ be a surjective homomorphism, $P_1, P_2 \leq G$ be $\mathbf{Z}$-equivalent subgroups, and $\Gamma_i = \psi^{-1}\left(P_i\right)$. Then for any $\mathbf{Z}[\Gamma]$-module A and any nonnegative integer k, there is a compatible isomorphism $H^k\left(\Gamma_1, \mathrm{Res}^\Gamma_{\Gamma_1}(A)\right) \to H^k\left(\Gamma_2, \mathrm{Res}^\Gamma_{\Gamma_2}(A)\right)$*

By letting A be a trivial $\mathbb{Z}[\Gamma]$-module (meaning: any module on which $\mathbb{Z}[\Gamma]$ acts trivially) in lemma 3.2.3, one obtains

Corollary 3.2.1. *Let $\psi : \Gamma \to G$ be a surjective homomorphism, $P_1, P_2 \leq G$ be $\mathbf{Z}$-equivalent subgroups, and $\Gamma_i = \psi^{-1}\left(P_i\right)$. Then for any trivial $\mathbf{Z}[\Gamma]$-module A and any nonnegative integer k, there is a compatible isomorphism $H^k\left(\Gamma_1, A\right) \to H^k\left(\Gamma_2, A\right)$*

3.3 Proof of theorem 1

The main goal of this section is the following construction of arbitrarily large families of finite index subgroups of certain lattices that are pairwise non-isomorphic and pairwise **Z**-equivalent.

Throughout this section, for each $r \in \mathbf{N}$, we will denote the free group of rank r by F_r.

Proposition 3.3.1. *Let* G *be a simple Lie group that is not isogenous to* $\mathrm{SL}(2, \mathbf{R})$ *and let* $\Gamma \leq \mathrm{G}$ *be a lattice that is large and non-arithmetic. Then for each* $j \in \mathbf{N}$*, there exist finite index subgroups* $\Delta_1, \ldots, \Delta_j \leq \Gamma$ *such that*

1. *The subgroups* Δ_i *are pairwise non-isomorphic.*

2. *The subgroups* Δ_i *are pairwise* **Z***-equivalent.*

We note that this proposition holds when G is isogenous to $\mathrm{SL}(2, \mathbf{R})$ but with (a) changed to the conition that the subgroups Δ_i are pairwise non-conjugate in G.

Lemma 3.3.1. *If* Q *is a finite group that is minimally generated by* r_Q *elements, then* $|\mathrm{Hom}_{sur}\ (F_r, Q)| \geq |Q|^{r-r_Q}$ *for all* $r \geq r_Q$*.*

Proof. Given $r \geq r_Q$, let $X_r = \{x_1, \ldots, x_r\}$ and let $F_r = F(X_r)$ be the free group generated by X_r. We can view $F_{r_Q} \leq F_r$ by $F_{r_Q} = \left\langle x_1, \ldots, x_{r_Q} \right\rangle$. Fixing $\varphi \in \mathrm{Hom}_{\mathrm{sur}}\left(F_{r_Q}, Q\right)$, for each $q_{r_Q+1}, \ldots, q_r \in Q$, we define $\Phi : F_r \to Q$ to be the unique homomorphism induced by the function $f : X_r \to Q$ given by

$$f(x_j) = \begin{cases} \varphi(x_j), & j \leq r_Q, \\ q_j, & j > r_Q. \end{cases}$$

Since φ is surjective, the homomorphisms Φ are surjective and distinct for all distinct (as ordered sets) choices of $q_{r_Q+1}, \ldots, q_r$. Hence $|\mathrm{Hom}_{\mathrm{sur}}\ (F_r, Q)| \geq |Q|^{r-r_Q}$ □

We will make use of the following theorem of Hall **HALL-[EULERIANFUNCTIONSGROUP]1936**

Theorem 3.3.1. *Let* Q *be a non-abelian finite simple group and* Γ *be a finitely generated group. If* $\varphi_1, \ldots, \varphi_m \in \mathrm{Hom}_{sur}(\Gamma, Q)$ *and* $\varphi_i \neq \theta \circ \varphi_j$ *for all* $\theta \in \mathrm{Aut}(Q)$ *and all* $i \neq j$*, then* $\varphi_1 \times \cdots \times \varphi_m : \Gamma \to Q^m$ *is surjective.*

Proof of Proposition 3.3.1. We begin by setting $\mathcal{X}_r(Q) \overset{\text{def}}{=} \mathrm{Hom}_{\mathrm{sur}}\,(F_r, Q)\,/\,\mathrm{Aut}(Q)$ where the action of $\mathrm{Aut}(Q)$ on $\mathrm{Hom}_{\mathrm{sur}}\,(F_r, Q)$ is given by post-composition. By Lemma 4.], we see that $\beta_{r,Q} = |\mathcal{X}_r(Q)| \geq \alpha_Q^{-1}|Q|^{r-r_Q}$ where $\alpha_Q = |\,\mathrm{Aut}(Q)|$. For each equivalence class x in $\mathcal{X}_r(Q)$, we fix a representative $\varphi_x \in \mathrm{Hom}_{\mathrm{sur}}\,(F_r, Q)$. By Theorem 4.3, we have a surjective homomorphism $\Phi_r : F_r \to Q^{\beta_r Q}$ given by $\Phi_r = \prod_{x \in \mathcal{X}_r(Q)} \varphi_x$. Fixing $Q = \mathrm{PSL}\,(2, \mathbf{F}_{29})$ and setting $P_1, P_2 \leq Q$ to be the $\mathbf{Z}$-equivalent subgroups given by Scott [39], for each $m \in \mathbf{N}$ and $z = (z_i) = \{1,2\}^m$, we define $P_z \leq Q^m$ to be the subgroup $P_z \overset{\text{def}}{=} \prod_{i=1}^m P_{z_i}$. It follows that for any distinct $z, z' \in \{1,2\}^m$ that $P_z, P_{z'}$ are $\mathbf{Z}$-equivalent and non-conjugate in Q^m. In particular, Q^m has 2^m pairwise nonconjugate, pairwise $\mathbf{Z}$-equivalent subgroups.

Now, given a large, non-arithmetic lattice $\Gamma \leq \mathrm{G}$ and $j \in \mathbf{N}$, we must find finite index subgroups $\Delta_1, \ldots, \Delta_j \leq \Gamma$ that are pairwise non-isomorphic and pairwise $\mathbf{Z}$-equivalent. Since Γ is non-arithmetic, combining Mostow-Prasad (see **Mostow-[StrongRigidityLocally]1973** and **Prasad-[StrongRigidityQrank]1973**) and Margulis **Margulis-[DiscreteSubgroupsSemisimple]** there exists a constant $C_\Gamma \in \mathbf{N}$ such that if $\Delta \leq \Gamma$ is a finite index subgroup, there are at most C_Γ non-conjugate subgroups of Γ that are isomorphic to Δ as an abstract group. Explicitly, $C_\Gamma = [\mathrm{Comm}(\Gamma) : \Gamma]$ and so when $\Lambda \leq \Gamma$ is a finite index subgroup, we have $C_\Lambda = C_\Gamma[\Gamma : \Lambda]$. As Γ is also large, there exists a finite index subgroup $\Gamma_2 \leq \Gamma$ and a surjective homomorphism $\psi : \Gamma_2 \to F_2$. Given any $r \geq 3$, there exists a subgroup $F_r \leq F_2$ of index $r-1$ such that F_r is a free group of rank r. To see this, we first note that we have a surjective homomorphism $F_2 \to \mathbf{Z}$ given by sending $a = 1$ and $b = 0$, where $\{a, b\}$ is a free basis for F_2. We compose this surjection with the surjective homomorphism $\mathbf{Z} \to \mathbf{Z}/(r-1)\mathbf{Z}$ given by reduction modulo $r-1$. The kernel of the homomorphism $F_2 \to \mathbf{Z} \to \mathbf{Z}/(r-1)\mathbf{Z}$ has index $r-1$ in F_2. It follows by the Nielsen-Schreier theorem that this subgroup of F_2 is free and of rank r. Setting $\Gamma_r = \psi^{-1}(F_r)$, we see that there exists subgroups $\Gamma_r \leq \Gamma_2 \leq \Gamma$ and surjective homomorphisms $\psi_r : \Gamma_r \to F_r$ with $[\Gamma_2 : \Gamma_r] = r-1$. Now, for the given $j \in \mathbf{N}$, we select r such that $2^{\beta_{r,Q}} \geq j(r-1)C_{\Gamma_2}$. Note that this can be done since $\beta_{r,Q} \geq \alpha_Q^{-1}|Q|^{r-2}$ grows exponentially as a function of r whereas $(r-1)C_{\Gamma_2}$ only grows linearly as a function of r. By selection of Γ_r and r, we have the surjective homomorphism $\mu_r : \Gamma_r \to Q^{\beta_{r,Q}}$.

For each $z \in \{1,2\}^{\beta_r Q}$, we define $\Delta_z = \mu_r^{-1}(P_z)$ and note that the subgroups Δ_z are pairwise non-conjugate in Γ_r and are pairwise $\mathbf{Z}$-equivalent. There are $2^{\beta_{r,Q}}$ such subgroups and we know that for each Δ_z, there are at most C_{Γ_r} subgroups from this list that can be abstractly isomorphic to a fixed Δ_z. As $C_{\Gamma_r} = (r-1)C_{\Gamma_2}$ and $2^{\beta_{r,Q}} \geq j(r-1)C_{\Gamma_2}$, there is a subset of these subgroups of size at least j that are all pairwise non-isomorphic. $\square$

We are now prepared to prove theorem 1.

Proof of theorem 1. By theorem 9.2 in **Agol-[VirtualHakenConjecture]2013**, every closed hyperbolic 3-manifold is large. In higher dimensions, via the construction of Gromov-Piatetski-Shapiro, there exists infinitely many commensurability classes of complete, finite volume hyperbolic n-manifolds that are both non-arithmetic and large. We can apply Proposition 3.3.1 to any manifold M in the above classes. We have opted to only write out the case when M is a closed hyperbolic n-manifold as the complex hyperbolic setting is logically identical.

Given $j \in \mathbf{N}$, $n \geq 3$, and a closed hyperbolic n-manifold M which is non-arithmetic and large, we can apply Proposition 3.3.1 with $\Gamma = \pi_1(M)$. We obtain j pairwise non-isomorphic, finite index subgroup $\Delta_1, \ldots, \Delta_j$ that are $\mathbf{Z}$-equivalent. By Corollary 3.5, for any abelian group A endowed with a trivial $\mathbf{Z}[\Gamma]$-module structure, we obtain compatible isomorphisms between the cohomology groups $H^k(\Delta_i, A)$ and $H^k(\Delta_{i'}, A)$ for all k and all i, i'. Since M is aspherical, M is a $K(\Gamma, 1)$ for Γ. Setting M_i to be the associated finite covers corresponding to Δ_i, we see that M_i is a $K(\Delta_i, 1)$ for all i. In particular, we have that $H^k(M_i, A)$ and $H^k(\Delta_i, A)$ are compatibly isomorphic; the compatibility of the isomorphisms between $H^k(\Delta_i, A)$ and $H^k(\Delta_{i'}, A)$ produce compatible isomorphisms between the cohomology groups $H^k(M_i, A)$ and $H^k(M_{i'}, A)$. As the groups $\Delta_i, \Delta_{i'}$ are not isomorphic, by Mostow-Prasad rigidity the manifolds $M_i, M_{i'}$ are not isometric. Taking $A = \mathbf{Z}$ produces (3) of Theorem 1. The proof of Theorem 1 is completed by noting that $\mathbf{Z}$-equivalence implies $\mathbf{Q}$-equivalence and $\mathbf{Q}$-equivalence implies the manifolds $M_i, M_{i'}$ satisfy (1) and (2) by **Sunada-[RiemannianCoveringsIsospectral]1985**. $\square$

4. SPECTRAL RIGIDITY

In preparation for the main theorem of this chapter, we first establish some notations and definitions.

4.1 Some basic hyperbolic geometry

We take the upper half plane

$$\mathfrak{H} = \{x + iy \in \mathbb{C} : y > 0\}$$

equipped with the Riemannian metric

$$g_{\text{hyp}} = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}$$

is a model for the hyperbolic plane: the unique simply connected, complete Riemannian manifold with scalar curvature -1. The invariant volume element for this metric is given by

$$\mathrm{d}\,\mathrm{vol}_{g_{\text{hyp}}} = \frac{\mathrm{d}x\,\mathrm{d}y}{y^2},$$

and the Laplace operator is

$$\Delta_{g_{\text{hyp}}} = \frac{1}{y^2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

The group

$$\mathrm{SL}_2(\mathbb{R}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1\right\}$$

acts isometrically on $(\mathfrak{H}, \mathrm{d}s^2)$ via linear fractional transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az + b}{cz + d}$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$. This action factors through $\mathrm{PSL}_2(\mathbb{R})$ which can be identified with the full group of orientation preserving isometries of $(\mathfrak{H}, \mathrm{d}s^2)$.

The **translation length** of an element $g \in \mathrm{SL}_2(\mathbb{R})$ is

$$\tau(g) = \inf\{\mathrm{dist}(gx, x) : x \in \mathfrak{H}\}. \tag{4.1}$$

Recalling the trichotomy of proposition 2.5.3 for elements of $\mathrm{SL}_2(\mathbb{R})$ we classify isometries of $\mathfrak{H}^2$ as follows:

A noncentral element $g \in \mathrm{SL}_2(\mathbb{R})$ is

Hyperbolic if $\tau(g) > 0$

Elliptic if $\tau(g) = 0$ and the function $x \mapsto d(gx, x)$ attains its minimum in $\mathfrak{H}^2$, and

Parabolic if $\tau(g) = 0$ but $\mathrm{dist}(gx, x) > 0$ for all $x \in \mathfrak{H}$.

If Γ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{R})$ then the quotient $\Gamma\backslash\mathfrak{H}^2$ is a hyperbolic 2-orbifold. When Γ is a torsion free uniform lattice in $\mathrm{SL}_2(\mathbb{R})$, the quotient $\Gamma\backslash\mathfrak{H}^2$ is in fact a closed hyperbolic surface, and each noncentral element is hyperbolic. In this case, there is a bijection between the conjugacy classes in Γ and closed geodesics in $\Gamma\backslash\mathfrak{H}^2$. If $\gamma \subset \Gamma$ is a such a class, and $g \in \gamma$ is a representative, then the length $\ell(\gamma)$ of γ is $\tau(g)$. As g is hyperbolic, modulo ± 1, it is conjugate to a unique matrix of the form $\left(\begin{smallmatrix} \lambda_g & \\ & \lambda_g^{-1} \end{smallmatrix}\right)$ where $\lambda_g \in \mathbb{R}_{>0}$ and $\lambda_g > \lambda_g^{-1}$. In this case, the translation length $\tau(g)$ is related to λ_g by the formula

$$\tau(g) = 2\log\lambda_g, \tag{4.2}$$

which in turn, can be expressed in terms of $\mathrm{tr}(g) = \lambda_g + \lambda_g^{-1}$. Indeed, $\{\lambda_g^{\pm 1}\}$ are the roots of the characterstic polynomial $x^2 - \mathrm{tr}\, gx + 1$ so that $\lambda_g^{\pm 1} = \frac{\mathrm{tr}\, g \pm \sqrt{\mathrm{tr}(g)^2 - 4}}{2}$ so that

$$|\mathrm{tr}\, g| = e^{\tau(g)/2} + e^{-\tau(g)/2} = 2\cosh\tau(g)/2. \tag{4.3}$$

A converse to this construction is given by the following

Proposition 4.1.1 (Uniformization). *Let (M, g) be a smooth compact surface with constant scalar curvature -1. Then there exists a cocompact lattice $\Gamma < \mathrm{SL}_2(\mathbb{R})$ which is torsion free modulo ± 1, and consists only of hyperbolic elements such that (M, g) is isometric to $\Gamma\backslash\mathfrak{H}$. Up to conjugation in $\mathrm{SL}_2(\mathbb{R})$ the lattice Γ is uniquely determined by the Riemannian metric on M.*

4.2 Arithmetic hyperbolic 2- and 3-manifolds

Let A be a quaternion algebra over a number field k, where k has r_1 real and r_2 complex places. Note, then, that $r_1 + 2r_2 = n = \dim_{\mathbb{Q}}(k)$. Let $\sigma_1, \dots \sigma_n$ denote the distinct embedings of k into $\mathbb{C}$, and write k_{ν_i} for the completion of k with respect to the archimedian place ν_i corresponding σ_i. We arrange the indices so that $k_{\nu_i} \cong \mathbb{R}$ for $1 \leq i \leq r_1$ while $k_{\nu_i} \cong \mathbb{C}$ for $r_1 + 1 \leq i \leq n$. Over each real place ν, one has

$$A_\nu \cong \begin{cases} M(2, \mathbb{R}) \text{ if } \nu \notin \mathrm{Ram}_\infty(A) \\ \mathcal{H} \text{ if } \nu \in \mathrm{Ram}_\infty(A) \end{cases} \tag{4.4}$$

where $\mathcal{H}$ is the Hamiltonian quaternions, which is the unique quaternion division algebra over $\mathbb{R}$. Over each complex place ν, one has $A_\nu \cong M(2, \mathbb{C})$. The situation is summarized in the following

Proposition 4.2.1. Maclachlan.Reid-[ArithmeticHyperbolic3Manifolds]2003 *If A is ramified at s real places, then*

$$A \otimes_{\mathbb{Q}} \mathbb{R} \cong s\mathcal{H} \oplus (r_1 - s)M(2, \mathbb{R}) \oplus r_2 M(2, \mathbb{C}). \tag{4.5}$$

For each $i \leq n$, let ρ_i denote the composition of the natural inclusion $A \to A \otimes_{\mathbb{Q}} \mathbb{R}$ with the projection onto the i-th factor in the decomposition 4.5. For $i \leq r_1$, the reduced norm and trace maps are related by

$$\mathrm{trd} \circ \rho_i = \sigma_i \circ \mathrm{trd}, \quad \mathrm{nrd} \circ \rho_i = \sigma_i \circ \mathrm{nrd}, \tag{4.6}$$

while for $i > r_1$ the same is true up to a possible twist by complex conjugation. In particular, it follows that the image of the kernel A^1 under ρ_i of the reduced norm map $\mathrm{nrd} : A^\times \to k^\times$ lies in the semisimple Lie group $\mathrm{SL}_2(\mathbb{R})$ if $i \le r_1$ and A is unramified over ν_i, and in $\mathrm{SL}_2(\mathbb{C})$ if $i > r_1$.

The cases of principal interest to this paper are when either

- $r_2 = 0$ and $s = r_1 - 1$ so that k is totally real, and A is unramified over exactly one real place of k, or
- $r_1 = s$ and $r_2 = 1$ so that k is ramified over all of its real places, and has a unique complex place, over which A is necessarily unramified.

For reasons that will become apparent momentarily, we refer to such a quaternion algebra A as **Fuchsian type** (F-t) or **Kleinian type** (K-t), respectively.

In any case, so long as there exists at least one archimedean place $\nu \in \Omega_k$ over which A is unramified, the composition of the natural inclusion $A \to A \otimes_{\mathbb{Q}} \mathbb{R}$ of proposition 4.2.1 with the projection onto the unramified factors we obtain an embedding

$$\psi : A \to (r_1 - s)\,\mathrm{M}_2(\mathbb{R}) \oplus r_2\,\mathrm{M}_2(\mathbb{C}). \tag{4.7}$$

and any other such embedding will differ from ψ by a conjugation in $\mathrm{GL}_2(\mathbb{R})^{r_1-s} \oplus \mathrm{GL}_2(\mathbb{C})^{r_2}$.

Recall from section 2.3 that an order $\mathcal{O}$ in a quaternion algebra A over a field k with ring of integers R is a subring containing 1, which is finitely generated as a module over R, and which generates A as a k-vectorspace. Orders are natural analogues of rings of integers in noncommutative algebras, and serve as an essential source of locally symmetric spaces via the following observation:

Proposition 4.2.2. Maclachlan.Reid-[ArithmeticHyperbolic3Manifolds]2003 *Let $\mathcal{O}$ be an order in a quaternion algebra A over a number field k, such that A is unramified over at least one nonarchimedean place. Set $\mathcal{O}^1 = \mathcal{O} \cap A^1$. Then, with the embedding ψ given in 4.7, the image $\psi(\mathcal{O}^1)$ is an irreducibles lattice in $\mathrm{SL}_2(\mathbb{R})^{r_1-s} \times \mathrm{SL}_2(\mathbb{C})^{r_2}$. If, in addition, A is a quaternion division algebra, then this lattice is cocompact.*

4.2.1 Arithmetic Fuchsian and Kleinian groups

When the quaternion algebra A is a Fuchsian or Kleinian type (cf. 4.2), the image $\Gamma_{\mathcal{O}} := \psi(\mathcal{O}^1)$ is a lattice in $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$, respectively. The group $\Gamma_{\mathcal{O}}$ is called an arithmetdic Fuchsian or Kleinian group, respectively, and the associated locally symmetric space is a finite volume hyperbolic 2- or 3-orbifold. More generally, we say that a lattice Λ in $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$ is **arithmetic** if there exists a quaternion algebra A of Fuchsian or Kleinian type, and an order $\mathcal{O}$ in A, such that Λ is commesnurable, in the wide sense, with $\psi(\mathcal{O}^1)$, where ψ is any embedding of A in $\mathrm{M}_2(\mathbb{R})$ or $\mathrm{M}_2(\mathbb{C})$ as in 4.7. We say that an arithemtic Fuchsian or Kleinian lattice Λ is **derived** from a quaternion algebra A if there is an order $\mathcal{O}$ in A such that Λ is conjugate to a subgroup of $\Gamma_{\mathcal{O}}$, rather than merely commensurable with it.

Remark 4.2.1. We remark briefly that this definition of arithmeticity is inherently invariant up to conjugation in $\mathrm{GL}(2, \mathbb{R})$ and $\mathrm{GL}(2, \mathbb{C})$. This, in turn, is tantamount to arithemticity being a property intrinsic to the underlying Riemannian orbifold. We shall later see, and will make essential use of the fact that arithmeticity in dimensions 2 and 3 is actually a spectral invariant.

4.2.2 Congruence lattices

A large, albeit not exhaustive, supply of arithmetic Fuchsian and Kleinian groups arise from the family of congruence subgroups, which are defined as follows.

Fix a quaternion algebra A of Fuchsian or Kleinian type over a field k with ring of integers R, and an R-order $\mathcal{O}$ in A. For any integral ideal $\mathfrak{a} \subset \mathcal{O}$ in k, the set

$$\mathfrak{a}\mathcal{O} = \{ax : a \in \mathfrak{a}, x \in \mathcal{O}\} \tag{4.8}$$

is a two sided $\mathcal{O}$-ideal. Hence, the reduction map $\pi_{\mathfrak{a}} : \mathcal{O} \to \mathcal{O}/\mathfrak{a}\mathcal{O}$ is an epimorphism of R-algebras, with the latter being a finite algebra over $R/\mathfrak{a}$. This induces an epimorphism of groups $\mathcal{O}^1 \to (\mathcal{O}/\mathfrak{a}\mathcal{O})^1$. The **principal congruence subgroup of level $\mathfrak{a}$** is defined as the kernel of this map, i.e.

$$\mathcal{O}^1(\mathfrak{a}) := \mathcal{O}^1 \cap (1 + \mathfrak{a}\mathcal{O}) = \{x \in \mathcal{O}^1 : x \equiv 1 \mod \mathfrak{a}\mathcal{O}\}. \tag{4.9}$$

With ψ an embedding as in 4.7, we write $\Gamma_{\mathcal{O}}(\mathfrak{a}) := \psi(\mathcal{O}^1(\mathfrak{a}))$ for its image in $\mathrm{SL}(2,\mathbb{R})$ or $\mathrm{SL}(2,\mathbb{C})$, and $X_{\mathcal{O}}(\mathfrak{a})$ for the corresponding locally symmetric space.

The inclusion $\Gamma_{\mathcal{O}}(\mathfrak{a}) \to \Gamma_{\mathcal{O}}$ as a normal subgroup induces a regular cover $X_{\mathcal{O}}(\mathfrak{a}) \to X_{\mathcal{O}}$ with deck group isomorphic to $\Gamma_{\mathcal{O}}(\mathfrak{a})\backslash\Gamma_{\mathcal{O}} \cong (\mathcal{O}/\mathfrak{a}\mathcal{O})^1$.

Definition 4.2.1. *We say A is of*

- ***Fuchsian-type*** *if: k is totally real, and A is unramified over exactly one real place ν_o. In thise case, we use ν_o to identify k with a subfield of $\mathbb{R}$, and replace ν_o-subscripts with $\mathbb{R}$. Thus if A is of fuchsian type, one has*

$$G_{\mathbb{R}} \approx \mathrm{GL}_2(\mathbb{R}), \quad G_{\mathbb{R}} \approx \mathrm{SL}_2(\mathbb{R}), \quad \bar{G}_{\mathbb{R}} \approx \mathrm{PGL}_2(\mathbb{R}) \tag{4.10}$$

- ***Kleinian-type*** *if: k has exactly one complex place ν_o, and A is ramified over all real places. In this case, use ν_o to identify k with a subfield of $\mathbb{C}$, and replace ν_o-subscripts with $\mathbb{C}$. Thus if A is of Kleinian type, one has*

$$G_{\mathbb{C}} \approx \mathrm{GL}_2(\mathbb{C}), \quad G_{\mathbb{C}} \approx \mathrm{SL}_2(\mathbb{C}), \quad \bar{G}_{\mathbb{C}} \approx \mathrm{PGL}_2(\mathbb{C}) \tag{4.11}$$

If A is a quaternion algebra of Fuchsian or Kleinian type, we pick once and for all the isomorphisms in 4.10 and 4.11 respectively.

If A is of Fuchsian (resp. Kleinian) type then for any order $\mathcal{O}$, $\Gamma_{\mathcal{O}}$ is a lattice in $G_{\mathbb{R}} \approx \mathrm{SL}_2(\mathbb{R})$ (reps. in $G_{\mathbb{C}} \approx \mathrm{SL}_2(\mathbb{R})$). We call $\Gamma_{\mathcal{O}}$ the **arithmetic lattice associated to $\mathcal{O}$**. Let $\mathbb{H}$ be hyperbolic 2- or 3-space, according to whether A is Fuchsian or Kleinian. Then $\Gamma_{\mathcal{O}}$ acts on $\mathbb{H}$ properly discontinuously by isometries, and we write $X(\Gamma_{\mathcal{O}})$ for the quotient orbifold $\Gamma_{\mathcal{O}}\backslash\mathbb{H}$.

Definition 4.2.2. .

1. *A subgroup Λ of $\mathrm{SL}_2(\mathbb{R})$ (resp. $\mathrm{SL}_2(\mathbb{C})$) is* ***arithmetic*** *if there exists a quaternion algebra A over a number field k of Fuchsian (resp. Kleinian) type and an order $\mathcal{O}$ in A such that Λ is commensurable (in the wide sense) with the arithmetic lattice $\Gamma_{\mathcal{O}}$ associated to $\mathcal{O}$.*

2. *We say that Λ is* ***derived from*** *$\mathcal{O}$ if it is conjugate in $\mathrm{SL}_2(\mathbb{R})$ (resp. $\mathrm{SL}_2(\mathbb{C})$) to a subgroup of $\Gamma_{\mathcal{O}}$.*

3. *Say that Λ is* ***derived from*** *A if it is derived from some order in A. If M is a hyperbolic –or- manifold, we say that M is arithmetic (resp. derived from an order $\mathcal{O}$ or a quaternion algebra A) if there exists an arithmetic lattice Λ such that M is isometric to $\Lambda\backslash\mathbb{H}$ for some arithmetic lattfice Λ (resp. arithmetic lattice derived from $\mathcal{O}$ or A).*

4.2.3 Invariants of arithmetic Fuchsian and Kleinian groups

Let Λ be a subgroup of $\mathrm{SL}_2(\mathbb{C})$. Write $\operatorname{tr}\Lambda$ for the set $\{\operatorname{tr} g : g \in \Lambda\} \subset \mathbb{C}$ and $\mathbb{Q}(\operatorname{tr}\Lambda)$ for the **trace field of** Λ, the subfield of $\mathbb{C}$ generated by $\operatorname{tr}\Lambda$. We write

$$A_0\Lambda = \{\sum_i a_i g_i : a_i \in \mathbb{Q}\operatorname{tr}\Lambda, g_i \in \Lambda\},$$

for the subring of $\mathrm{M}_2(\mathbb{C})$ generated by Λ as an algebra over $\mathbb{Q}\operatorname{tr}\Lambda$. Then we have

Proposition 4.2.3. *Suppose Λ is a finitely generated, nonelementary subgroup of $\mathrm{SL}_2(\mathbb{C})$. Then $A_0\Lambda$ is a quaternion algebra over $\mathbb{Q}\operatorname{tr}\Gamma$. Furthermore*

1. *If Λ is a lattice in $\mathrm{SL}_2(\mathbb{C})$ or an arithmetic lattice in $\mathrm{SL}_2(\mathbb{R})$ then $\mathbb{Q}\operatorname{tr}\Lambda$ is a number field.*

2. *Λ is conjugate to a subgroup of $\mathrm{SL}_2(\mathbb{R})$ if and only if $\mathbb{Q}\operatorname{tr}\Lambda$ is contained $\mathbb{R} \subset \mathbb{C}$.*

3. *If Λ is an arithmetic lattice in $\mathrm{SL}_2(\mathbb{C})$ or $\mathrm{SL}_2(\mathbb{R})$, then Λ is derived from $A_0\Lambda$ if and only if $\operatorname{tr}\Lambda$ consists of algebraic integers in $\mathbb{Q}\operatorname{tr}\Lambda$.*

The following basic facts will be used in what follows.

Proposition 4.2.4. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.*

1. *If $\mathcal{O}, \mathcal{O}'$ are two orders in a quaternion algebra A, then $\Gamma_{\mathcal{O}}$ and $\Gamma_{\mathcal{O}'}$ are commensurable in $G = A^1$. Consequently, the property of being commensurable in $G_{\mathbb{K}}$ to an order in A depends only on A.*

2. *If $\mathcal{O}, \mathcal{O}'$ are orders in quaternion algebras A, A' over number fields k, k' with $A_{\mathbb{K}} = A'_{\mathbb{K}}$ such that $\Gamma_{\mathcal{O}}$ is commensurable to $\Gamma'_{\mathcal{O}}$ in $A^1_{\mathbb{K}} = {A^1}^1_{\mathbb{K}}$ then $k = k'$ and $A = A'$. Consequently, the quaternion algebra is an invariant of the commensurability class of an arithmetic lattice in $\mathrm{SL}_2(\mathbb{K})$.*

3. *The quaternion algebra of an arithmetic lattice in $\mathrm{SL}_2(\mathbb{K})$ is a complete invariant of its commensurability class.*

4. *An arithmetic lattice is cocompact if and only if its quaternion algebra is a division algebra.*

The following theorem asserts that arithmeticity of a Fuchsian or Kleianian group can be detected by the set of traces of its elements.

Proposition 4.2.5 (Takeuchi **Takeuchi-[CharacterizationArithmeticFuchsian]1975**). *Let Γ be a Fuchsian or Kleinian group of the first kind. Then Γ is an arithmetic Fuchsian or Kleinian group derived from a quaternion algebra if and only if Γ satisfies the following conditions*

1. *The subfield k of $\mathbb{C}$, generated over $\mathbb{Q}$ by the traces of elements of Γ, has finite degree over $\mathbb{Q}$*

2. $\mathrm{tr}(\Gamma)$ *is contained in the ring of integers R_k of k*

3. *For any isomorphism $\varphi : k \to \mathbb{C}$ such that $\varphi \neq \mathrm{id}$, the set $\varphi(\mathrm{tr}(\Gamma))$ is bounded in $\mathbb{C}$.*

We will also make use of the following theorem

Proposition 4.2.6 (**Reid-[IsospectralityCommensurabilityArithmetic]1992**). *Let M_1 and M_2 be isospectral arithmetic hyperbolic 2 or 3 manifolds. Then M_1 and M_2 are commensurable.*

4.3 Main Theorem

The main theorem of this section is the following:

Theorem 2. *Let A be a quaternion algebra over a number field k of Fuchsian (resp. Kleinian) type. Let $\mathcal{O}$ be a maximal order in A, and $\mathfrak{a}$ be an integral ideal in k. Let $\Gamma_{\mathcal{O}}(\mathfrak{a})$ be the principal congruence arithmetic lattice in* $\mathrm{SL}_2(\mathbb{R})$ *(resp.* $\mathrm{SL}_2(\mathbb{C})$*) of level $\mathfrak{a}$, and let $X(\Gamma_{\mathcal{O}}(\mathfrak{a}))$ be the associated hyperbolic 2-orbifold $\Gamma_{\mathcal{O}}(\mathfrak{a})\backslash\mathbb{H}^2$ (resp. hyperbolic 3-orbifold $\Gamma_{\mathcal{O}}(\mathfrak{a})\backslash\mathbb{H}^3$). Suppose that*

1. *A has type number 1, and*

2. *$\mathfrak{a}$ is not divisible by any prime over which A is ramified.*

Then $X(\Gamma_{\mathcal{O}}(\mathfrak{a}))$ is absolutely spectrally rigid.

Proof. First suppose that A is of Fuchsian type, and let (M,g) be a closed Riemannian manifold which is isospectral to $X(\Gamma_0(\mathfrak{a}))$. By corollary 2.1.1 we find that $\dim M = \dim X(\Gamma_0(\mathfrak{a})) = 2$. As $X(\Gamma_0(\mathfrak{a}))$ has constant scalar curvature -1, we may conclude by proposition 2.1.3 that so too does (M,g). By the Uniformization theorem of 4.1.1, there exist a cocompact Fuchsian lattice $\Lambda \leq \mathrm{SL}_2(\mathbb{R})$ such that (M,g) is isometric to $\Lambda\backslash\mathfrak{H}$. By the Selberg's trace formula, the equality of the Laplace spectra for $\Lambda\backslash\mathfrak{H}$ and $\Gamma_{\mathcal{O}}(\mathfrak{a})\backslash\mathfrak{H}$ is equivalent to the equality of their length spectra. From the relation 4.3 between the length of a geodesic and the trace of the corresponding hyperbolic translation, we conclude the equality of trace sets $\mathrm{tr}\,\Lambda = \mathrm{tr}\,\Gamma_{\mathcal{O}}(\mathfrak{a})$. Applying Takeuchi's classification of arithmetic Fuchsian groups of proposition 4.2.5 we find that Λ is an arithmetic Fuchsian lattice, and that it is *derived* from its invariant quaternion algebra. Applying 4.2.6, we find that Λ and $\Gamma_{\mathcal{O}}(\mathfrak{a})$ are furthermore commensurable. Thus there exists some maximal order $\mathcal{O}'$ in A such that Λ is conjugate in $\mathrm{SL}_2(\mathbb{R})$ to a finite index subgroup of $\Gamma_{\mathcal{O}'}$. By assumption, there is a unique $A^\times$ conjugacy class of maximal orders in A, so we can in fact take $\mathcal{O}' = \mathcal{O}$.

To summarize, if (M, g) is any closed Riemannian manifold which is isospectral to $\Gamma_{\mathcal{O}}(\mathfrak{a})\backslash\mathfrak{H}$, then (M, g) is isometric to $\Lambda\backslash\mathfrak{H}$ for some finite index subgroup Λ of $\Gamma_{\mathcal{O}} = \mathcal{O}^1 \leq A^1 \leq \mathrm{SL}_2(\mathbb{R})$. Furthermore, observe that

$$\mathrm{vol}(M, g) = \mathrm{vol}(\Lambda\backslash\mathfrak{H}) = |\Gamma_{\mathcal{O}} : \Lambda| \,\mathrm{vol}(\Gamma_{\mathcal{O}}\backslash\mathfrak{H}) \tag{4.12}$$

and by 2.1.1 we

$$\mathrm{vol}(M, g) = \mathrm{vol}(\Gamma_{\mathcal{O}}(\mathfrak{a})\backslash\mathfrak{H}) = |\Gamma_{\mathcal{O}} : \Gamma_{\mathcal{O}}(\mathfrak{a})| \,\mathrm{vol}(\Gamma_{\mathcal{O}}\backslash\mathfrak{H}) \tag{4.13}$$

so that $|\Gamma_{\mathcal{O}} : \Lambda| = |\Gamma_{\mathcal{O}} : \Gamma_{\mathcal{O}}(\mathfrak{a})|$.

To complete the proof of theorem 2, we will show the following

Lemma 4.3.1. *Let k be a number field with ring of integers R. Let A be a quaternion algebra over k, $\mathcal{O}$ be a maximal order in A, and $\mathfrak{a}$ be an ideal in R. If H is any subgroup of $\mathcal{O}^1$ such that such that $\mathrm{tr}\, H \subseteq \mathrm{tr}\, \Gamma_{\mathcal{O}}(\mathfrak{a})$, then in fact H is conjugate, by an element of $\widetilde{G}(k) = A^\times$, to a subgroup of $\Gamma_{\mathcal{O}}(\mathfrak{a}) = (1 + \mathfrak{a}\mathcal{O}) \cap \mathcal{O}^1$.*

To this end, first we identify the set $\mathrm{tr}(\Gamma_{\mathcal{O}}(\mathfrak{a})) \subset R$:

Lemma 4.3.2. *Let A be a quaternion algebra over a number field or a nonarchimedean local field k with ring of integers R. Let $\mathcal{O}$ be a maximal order in A and $\mathfrak{a}$ an ideal in R. Then*

$$\mathrm{tr}\, \Gamma_{\mathcal{O}}(\mathfrak{a}) = \{t : t^2 - 4 \equiv 0 \mod \mathfrak{a}^2\}. \tag{4.14}$$

Proof. First suppose k is a nonarchimedean local field, so that $\mathfrak{a} = \mathfrak{p}^n$ for some $n \geq 0$ where $\mathfrak{p}$ is the unique maximal ideal of R. Pick a uniformizer $\varpi \in R$ for $\mathfrak{p}$.

If A is unramified over $\mathfrak{p}$, then there exists an isomorphism $A \cong \mathrm{M}_2(k)$ and any two such isomorphisms differ by an inner automorphism by $A^\times \cong \mathrm{GL}_2(k)$. Choose such an isomorphism which induces an isomorphism of $\mathcal{O}$ with $\mathrm{M}_2(R)$, and in turn $\mathcal{O}^1$ with $\mathrm{SL}_2(R)$. Suppose now that $g \in \mathrm{SL}_2(R)$ is of the form $g = 1 + \varpi^n h$ for some $h \in \mathrm{M}_2(R)$. Then

$$\begin{aligned}\det g &= \mathrm{nrd}(1 + \varpi^n h)\\ &= 1 + \varpi^n \operatorname{tr} h + \varpi^{2n} \det h,\end{aligned}$$

and since $\det g = 1$, we find $\operatorname{tr} h + \varpi^n \det h = 0$, and in turn

$$(\operatorname{tr} g - 2)\varpi^{-2n} = \operatorname{tr}(h)\varpi^{-n} = \det h \in R,$$

so that $\operatorname{tr} g = 2 \mod \varpi^{2n} R$ as claimed.

Conversely, suppose $t \in R$ takes the form $2 \pm \varpi^{2n} s$ for some $s \in R$. Then the matrix $g = \begin{pmatrix} 1 + s\varpi^2 & \varpi^n \\ s\varpi^n & 1 \end{pmatrix}$ satisfies $\operatorname{tr} g = t$, $\det g = 1$, and $g \equiv \pm \mathrm{id} \mod \mathfrak{p}^n$.

Now suppose k is a number field. Then the claim follows from a routine application of local to global principle.

$\square$

Lemma 4.3.3. *Let A be a quaternion algebra over a number field k with ring of integers R, and let $\mathcal{O}$ be a maximal order in A. Let $\mathfrak{a}$ be an ideal of R which is coprime to the discriminant of A, and suppose H is a subgroup of $\mathcal{O}^1$ such that $\operatorname{tr} H \subseteq \operatorname{tr} \Gamma_{\mathcal{O}}(\mathfrak{a})$. Then there exists an $\alpha \in A^\times$ such that $\alpha H \alpha^{-1} \leq \mathcal{O}^1(\mathfrak{a})$.*

Proof. Given the description of $\operatorname{tr} \Gamma^1_{\mathcal{O}}(\mathfrak{a})$ of lemma 4.3.2, we see that if $\mathfrak{p}$ is a prime in R with $\mathfrak{p}^{e_\mathfrak{p}} || \mathfrak{a}$ then for all $h \in H$, one has $\delta_h = \operatorname{trd} h^2 - 4 \equiv 0 \mod \mathfrak{p}^{2e_\mathfrak{p}}$, and by assumption A is unramified over $\mathfrak{p}$, so that $\mathcal{O}^1_\mathfrak{p} \cong \mathrm{SL}_2(R_\mathfrak{p})$ in $A^1_\mathfrak{p} \,\mathrm{SL}_2(k_\mathfrak{p})$. By proposition 2.5.8, there exists an $\alpha_\mathfrak{p} \in A^\times_\mathfrak{p}$ such that

$$\alpha_\mathfrak{p} H \alpha_\mathfrak{p}^{-1} \subset O^1_\mathfrak{p}(\mathfrak{a}) \cong \mathrm{SL}_2(R_\mathfrak{p}, \mathfrak{p}^{e_\mathfrak{p}}) \tag{4.15}$$

Furthermore, adjusting $\alpha_\mathfrak{p}$ by an element of $\mathrm{GL}_2(R_\mathfrak{p})$ does not effect equation 4.15 as the latter normalizes $\mathrm{SL}_2(R_\mathfrak{p}, \mathfrak{p}^{e_\mathfrak{p}})$.

Let α' be the element of $A^\times_\mathcal{A}$ which is $\alpha_\mathfrak{p}$ at each prime $\mathfrak{p}$ dividing $\mathfrak{a}$ and 1 at all other places. Then α' is a well-defined element of $A^\times_\mathcal{A}$ by the strong approximation theorem 2.4.1 there exists a representative $\alpha \in A^\times$ of the coset $\alpha'\hat{\mathcal{O}}^1 \in A^1_\mathcal{A}$ where $\hat{\mathcal{O}}$ is the adelic completion of $\mathcal{O}$. Then $\alpha H \alpha^{-1} \subset \mathcal{O}^1(\mathfrak{a})$ as desired.

□

To conclude the proof of the main theorem, we observe that Λ is conjugate to a subgroup of $\Gamma_\mathcal{O}(\mathfrak{a})$ by 4.3.3, while $|\Gamma_\mathcal{O} : \Gamma_\mathcal{O}(\mathfrak{a})| = |\Gamma_\mathcal{O} : \Lambda|$. Thus, Λ is in fact conjugate to $\Gamma_\mathcal{O}(\mathfrak{a})$, and the theorem follows. □

4.3.1 Applications

With theorem 2 proven, we now turn to applications. First, off noting that as in corollary 2.3.2, any quaternion algebra over a field with class number one is of type number one, we obtain the following corollary:

Corollary 4.3.1. *Let A be an indefinite quaternion algebra over $\mathbb{Q}$, of discriminant D, and $\mathcal{O}$ a maximal order in A. If N is an integer whihch is coprime to D, then the locally symmetric space associated to the congruence lattice $\Gamma_\mathcal{O}(N)$ is absolutely spectrally rigid.*

For the next, corollary let us recall a theorem due to Hurwitz:

Proposition 4.3.1. Vishne-[HurwitzQuaternionOrder]2011 *Let X be a compact Riemann surface of genus $g > 1$. Then the number of automorphisms of X is bounded above by $84(g-1)$.*

If X is a compact Riemann surface of genus $g > 1$ for which $|\operatorname{Aut}(X)| = 84(g-1)$, then X is known as a **Hurwitz surface**. In **Reid-[TracesLengthsAxes]2014** A. Reid asked whether a Hurwitz surface is determined by its spectrum. We partially answer this question in the affirmative, as follows: First, observe that, by definition, a Hurwitz surface X must be a regular cover of the unique minimal volume closed hyperbolic 2-orbifold, $\Gamma_{2,3,7}\backslash\mathfrak{H}$, where $\Gamma_{2,3,7}$ is the group of rotations in the corners of a hyperbolic triangle with angles $\pi/2$, $\pi/3$ $\pi/7$. It is known **Katz.Schaps.Vishne-[ExplicitComputationsHurwitz]2008** that

$\Gamma_{2,3,7}$ is an arithmetic Fuchsian group, and arises as the units of reduced norm 1 in the unique quaternion algebra over the totally real subfield K of $\mathbb{Q}[\zeta]$ where ζ is a primitive 7th root of 1. Thus, $K = \mathbb{Q}[\eta]$ where $\eta = \zeta + \zeta^{-1}$ satisfies the relation $\eta^3 + \eta^2 - 2\eta - 1 = 0$. There are three real places of K, sending η to any of the three real roots of the preceeding equation, namely the unique positive root $2\cos(2\pi/7) > 0$, and the two negative roots $2\cos(4\pi/7), 2\cos(6\pi/7) < 0$. The quaternion algebra over K which is ramified over the two negative real places, and unramfieid over all other places, has a maximal order with group of units of reduced norm 1, equal to $\Gamma_{2,3,7}$. Furthermore, as K has class number 1, this quaternion algebra has type number 1. Consequently, theorem 2 applies to say that any Hurwitz surface arising from a principal congruence subgroup of $\Gamma_{2,3,7}$ is spectrally rigid.

Corollary 4.3.2. *Let X be a principal congruence Hurwitz surface. Then X is spectrally rigid.*

VITA

Justin E. Katz was born in Michigan City, Indiana, on May 12, 1993. In 2011, he graduated from La Lumiere high school, in Laporte Indiana. The following August, he entered Reed College in Portland, Oregon, from which he graduated with a degree of Bachelor of Arts in Mathematics in May of 2015. The following August, he entered Purdue University in West Lafayette, Indiana, where he is currently a Ph.D. candidate in Mathematics.